W. Stein

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Visibility of Shafarevich-Tate Groups at Higher Level

Defn: An abelian variety is a projective group variety.

Example: The abelian varieties of dim. 1 are exactly the elliptic curves.

$y^2 = x^3 + ax + b$ (projective closure)

Let

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod N \right\}$$

cusp forms

$$S_2(\Gamma_0(N)) = \left\{ \begin{array}{l} \text{cusp forms of weight 2 for } \Gamma_0(N) \\ f : \mathfrak{h}^* \to \mathbb{C} \text{ such that } f(z)dz = f(\gamma z)d\gamma z \\ \text{holomorphic} \quad \forall \gamma \in \Gamma_0(N) \text{ and } f(\mathbb{Q}\cup\{\infty\}) = 0 \end{array} \right\}$$

$$\cong H^0(X_0(N), \Omega^1),$$

$$X_0(N) = \Gamma_0(N) \backslash \mathfrak{h}^*, \quad \text{genus } g$$

Hecke algebra

$$\mathbb{T} = \mathbb{Z}[T_1, T_2, T_3, \ldots] \leftarrow \text{commutative ring} \approx \mathbb{Z}^g \text{ as } \mathbb{Z}\text{-module}$$

Jacobian

$$J_0(N) = \mathrm{Jac}(X_0(N)) = \mathrm{Pic}^0(X_0(N))$$

Newform: $\mathbb{T}$-eigenform, $a_1 = 1$

$$f = \sum_{n=1}^{\infty} a_n q^n \in S_2(\Gamma_0(N))$$

simple modular abelian variety

$$A_f = J_0(N)[I_f]^0 \subseteq J_0(N)$$

$I_f = \mathrm{Ann}_{\mathbb{T}}(f)$

abelian variety over $\mathbb{Q}$ of $\dim = d = [\mathbb{Q}(\ldots a_n \ldots) : \mathbb{Q}]$ and $\mathrm{End}(A_f) \otimes \mathbb{Q} \cong \mathbb{Q}(\ldots a_n \ldots)$.

$\mathrm{Re}(s) > 3/2$

$$L(A_f, s) = \prod_\sigma L(f^\sigma, s) = \prod_\sigma \left(\sum \frac{a_n^\sigma}{n^s} \right)$$

Hasse-Weil L-function of A_f.

The Birch & Swinnerton-Dyer Conjecture

$A = A_f$ (there is also a conjecture for any ab var over number field or functionfield)

BSD-rank:

$$r = \operatorname{rank} A(\mathbb{Q}) \overset{\text{conj}}{=} \operatorname*{ord}_{s=1} L(A_f, s) \quad \left(\overset{?}{=} d \cdot \operatorname{ord}_{s=1} L(f,s) \right)$$

BSD-formula:

Tamagawa numbers $\#(A/A^{\circ})(\mathbb{F}_p)$ $\quad$ $\int_{A(\mathbb{R})} \omega$ $\quad$ |disc of heightpairing| $\quad$ Shafarevich-Tate group

$$\frac{L^{(r)}(A_f, 1)}{r!} = \frac{\left(\prod_{p \mid N} c_p\right) \cdot \Omega_A \cdot \mathrm{Reg}_A \cdot \# Ш(A)}{\# A(\mathbb{Q})_{tor} \cdot \# A^\vee(\mathbb{Q})_{tor}}$$

Facts:

Kolyvagin-Logachev (Gross-Zagier, etc...) also Kato

$\operatorname{ord}_{s=1} L(f,s) \le 1 \implies$ BSD-rank is true for A_f and $Ш(A)$ is finite.

The Shafarevich-Tate Group

Weil-Chatelet Group of classes of torsors X for A.

$$Ш(A) = \operatorname{Ker}\left(H^1(\mathbb{Q}, A) \longrightarrow \bigoplus_{\text{places } v} H^1(\mathbb{Q}_v, A) \right)$$

$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-cohomology

infinitely many elements of all order

- very mysterious
- finite? conjecturally...
- obstruction to local-to-global principal.

W. Stein
Vishigher

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Mazur's Notion of Visibility

Defn: Suppose $A \subset B$ inclusion of abelian varieties $/\mathbb{Q}$ (say)

$$\mathrm{Vis}_B H^1(\mathbb{Q},A) = \ker\left(H^1(\mathbb{Q},A) \longrightarrow H^1(\mathbb{Q},B)\right)$$

$$\begin{aligned}\mathrm{Vis}_B Ш(A) &= \ker\left(Ш(A) \longrightarrow Ш(B)\right)\\ &= Ш(A) \cap \mathrm{Vis}_B H^1(\mathbb{Q},A).\end{aligned}$$

Why "visible"? Let $C = B/A$ so

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

Long exact sequence:

0 → A(Q

$$\begin{array}{c} \in WC(A) \\ X = \pi^{-1}(x) \\ \cap \\ B \end{array}$$

$$x \mapsto c \in \mathrm{Vis}_B H^1(\mathbb{Q},A) \subset H^1(\mathbb{Q},A)$$

$$0 \longrightarrow A(\mathbb{Q}) \longrightarrow B(\mathbb{Q}) \longrightarrow C(\mathbb{Q}) \longrightarrow H^1(\mathbb{Q},A) \longrightarrow H^1(\mathbb{Q},B)$$

$\boxed{\text{Visible class}} \leftrightarrow$
- rational point on C.
- subvariety of B corresponding to element of WC

Prop: If $c \in H^1(\mathbb{Q},A)$ then there is a B such that

$$c \in \mathrm{Vis}_B H^1(\mathbb{Q},A).$$

[Proof: Use $B = \mathrm{Res}_{K/\mathbb{Q}} A$, where $c \mapsto 0 \in H^1(K,A)$.]

This B ~~often~~ is not modular. Too easy

Fun Application: First ever construction of A with

$$\# Ш(A) = p \cdot n^2 \text{ with } p \text{ odd}.$$

- Strategies for proving $\ell \mid c_p \Rightarrow \ell \mid \frac{L(E,1)}{\Omega_E}$ (mostly done)
- — Bounding Ш for rank 1 curves?

"Goal: Precisely connect visibility with extensive theory of congruences between modular forms, modular cohomology (e.g. Euler systems), etc."

~~In particular connect BSD rank ⟷ BSD formula~~

Defn: $c \in Ш(A_f)$ is visible of level M if there is B s.t.

$$A_f \hookrightarrow B$$

with B a quotient of some $J_0(M)$ and

$$c \in \mathrm{Vis}_B\, Ш(A_f).$$

Conjecture(-) Every element of $Ш(A_f)$ is visible of some level. (Infinitely many levels)

Challenge: Develop tools to improve understanding of conjecture, and say something conjectural about the levels at which c is modular.

Experiment (partly joint with A. Agashe and D. Jetchev)

1. Develop algorithms to compute good divisor and multiple of conjectural $\#Ш(A_f)$. ‖ a major component of my research
2. Compute divisor and multiple of all A_f of rank 0 and level ≤ 2333 (10,360 abelian varieties) 168 nontrivial odd divisor)
3. Compute all A_f of level ≤ 8000 (still missing 109 levels) found 95008 such A_f. (Cremona's tables have 29755 up to level 8000)
4. Define "weak modularity" of level M
5. Make table (see handout – explain) Include data from Cremona's table also...
6. ECDB?

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of level N

Defn: Say $Ш(A_f)[p]$ is weakly visible of level M if there is a newform $g \in S_2(\Gamma_0(M))$ in the table T such that

(a) $\mathrm{ord}_{s=1} L(g, s) \geq 2$

(b) For $\ell = 2, 3, 5, \ldots, 19$ and $\ell \nmid NM$,

$$\gcd(\mathrm{charpoly}(a_\ell(f)) \bmod p,\ \mathrm{charpoly}(a_\ell(g)) \bmod p) \neq 1$$

Motivation: $\begin{array}{ccc} B[p] & \subset & B \\ \cap & & \downarrow \\ A & \hookrightarrow & C \end{array}$ + hypo $\overset{\text{thm}}{\Longrightarrow} B(\mathbb{Q})/pB(\mathbb{Q}) \subset \mathrm{Vis}_B Ш(A)$. + BSD

Explain handout:

- 103 A_f's of level ≤ 2000 such that $\# Ш(A_f)$, divisible by odd prime.
- All but 11 have conjectural Ш weakly visible (and "many" are provably visible)!
- 28 A_f have Ш weakly visible only at a higher level.
- 64 A_f have Ш weakly visible at same level.

Heuristic Observation

Much Visible Ш $\Longrightarrow$ Many Elliptic Curves of Rank ≥ 2, W Stein (6)

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Cremona's data: All $E/\mathbb{Q}$ with conductor ≤ 25000.

rank 0: 42165 isogeny classes

rank 1: 53483 "

rank 2: 7509 " ← 7% of classes.

rank ≥3: 17 "

AWS Ellenberg story.

But maybe just "law of small numbers".

ECDB with Watkins:

All $E/\mathbb{Q}$ with $|\Delta| \leq 10^{12}$, $c_4 \leq 1.44 \cdot 10^{12}$, $N < 10^8$

"Analyzed" via student summer project (using Python/MySQL)
Baur Bektemirov

Up to 100,000:

rank 0: 40%

rank 1: 51%

rank 2: 9% ok

rank ≥3: ~0%

Up to 100 million (~135 million curves)

rank 0: 34%

rank 1: 48%

rank 2: 16% !!

rank ≥3: 2%

Next Computation:

Take $f \in S_2(389)$ with $\#C \in Ш(A_f^3)[5]$. Find all E in the ECDB such that $f \equiv f_E \pmod 5$ away from bad primes, and rank $E \geq 2$. What structure do the conductors of E have?