Elliptic Curves Over $F = \mathbb{Q}(\sqrt{5})$

William Stein (University of Washington)
in Chicago (UIC) at the Atkin Memorial Workshop

University of Washington

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Joint Work...

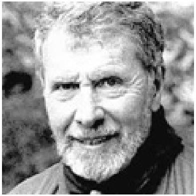
This talk represents joint work with Jonathan Bober, Alyson Deines, Joanna Gaski, Ariah Klages-Mundt, Benjamin LeVeque, R. Andrew Ohana, Ashwath Rabindranath, and Paul Sharaba.

Acknowledgement: John Cremona, Lassina Dembele, Noam Elkies, Tom Fisher, Richard Taylor, and John Voight for helpful conversations and data. I used Sage (`http://www.sagemath.org`) extensively.

Motivation

"The object of numerical computation is theoretical advance."

- Oliver Atkin



Contents

1: Tables

Source: These tables and much code were made at a summer REU[1] at University of Washington last summer.

See `https://github.com/williamstein/sqrt5`.

Remark: If E/F and $\sigma(\sqrt{5}) = -\sqrt{5}$, then E^σ is another curve over F. All of our tables *do* include *both* E and E^σ! We tried to avoid this redundancy but it caused too much confusion.

[1] Research Experience for Undergraduates

Counts of Curves over F up to Norm Conductor 1831

Table: Curves over $\mathbb{Q}(\sqrt{5})$

rank	**#isog**	**#isom**	**smallest** $\mathrm{Norm}(\mathfrak{n})$
0	745	2174	31
1	667	1192	199
2	2	2	1831
total	1414	3368	-

Number of Isogeny Classes over F up to Norm Conductor 1831

Table: Number of Isogeny classes of a given size

	size							
bound	1	2	3	4	6	8	10	**total**
199	2	21	3	20	8	9	1	64
1831	498	530	36	243	66	38	3	1414

Rank Data

Table: Counts of classes and curves with bounded norm conductors and specified ranks

	#isog				#isom			
	rank				rank			
bound	0	1	2	**total**	0	1	2	**total**
200	62	2	0	64	257	6	0	263
400	151	32	0	183	580	59	0	639
600	246	94	0	340	827	155	0	982
800	334	172	0	506	1085	285	0	1370
1000	395	237	0	632	1247	399	0	1646
1200	492	321	0	813	1484	551	0	2035
1400	574	411	0	985	1731	723	0	2454
1600	669	531	0	1200	1970	972	0	2942
1800	729	655	0	1384	2128	1178	0	3306
1831	745	667	2	1414	2174	1192	2	3368

Isogeny Degrees

Table: Isogeny degrees

degree	#isog	#isom	example curve	Norm($\mathfrak{n}$)
None	498	498	$[\varphi+1, 1, 1, 0, 0]$	991
2	652	2298	$[\varphi, -\varphi+1, 0, -4, 3\varphi-5]$	99
3	289	950	$[\varphi, -\varphi, \varphi, -2\varphi-2, 2\varphi+1]$	1004
5	65	158	$[1, 0, 0, -28, 272]$	900
7	19	38	$[0, \varphi+1, \varphi+1, \varphi-1, -3\varphi-3]$	1025

Torsion Subgroups of Elliptic Curves over F

(I don't trust this table.)

Table: Distribution of torsion subgroups up to norm conductor 1831

structure	**#isom**	**example curve**	Norm($\mathfrak{n}$)
1	296[2]	$[0, -1, 1, -8, -7]$	225
$\mathbb{Z}/2\mathbb{Z}$	1453	$[\varphi, -1, 0, -\varphi - 1, \varphi - 3]$	164
$\mathbb{Z}/3\mathbb{Z}$	202	$[1, 0, 1, -1, -2]$	100
$\mathbb{Z}/4\mathbb{Z}$	243	$[\varphi + 1, \varphi - 1, \varphi, 0, 0]$	79
$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	312	$[0, \varphi + 1, 0, \varphi, 0]$	256
$\mathbb{Z}/5\mathbb{Z}$	56	$[1, 1, 1, 22, -9]$	100
$\mathbb{Z}/6\mathbb{Z}$	183	$[1, \varphi, 1, \varphi - 1, 0]$	55
$\mathbb{Z}/7\mathbb{Z}$	13	$[0, \varphi - 1, \varphi + 1, 0, -\varphi]$	41
$\mathbb{Z}/8\mathbb{Z}$	21	$[1, \varphi + 1, \varphi, \varphi, 0]$	31
$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$	51	$[\varphi + 1, 0, 0, -4, -3\varphi - 2]$	99
$\mathbb{Z}/9\mathbb{Z}$	6	$[\varphi, -\varphi + 1, 1, -1, 0]$	76
$\mathbb{Z}/10\mathbb{Z}$	12	$[\varphi + 1, \varphi, \varphi, 0, 0]$	36
$\mathbb{Z}/12\mathbb{Z}$	6	$[\varphi, \varphi + 1, 0, 2\varphi - 3, -\varphi + 2]$	220
$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$	11	$[0, 1, 0, -1, 0]$	80
$\mathbb{Z}/15\mathbb{Z}$	1	$[1, 1, 1, -3, 1]$	100
$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$	2	$[1, 1, 1, -5, 2]$	45

[2] On the previous slide there were 498 with no isogenies, so this or that

Comparison: F versus $\mathbb{Q}$

Table: Distribution of torsion subgroups up to (norm) conductor 1831

structure	#isom over F	#isom over $\mathbb{Q}$
1	296 (*)	3603
$\mathbb{Z}/2\mathbb{Z}$	1453	4580
$\mathbb{Z}/3\mathbb{Z}$	202	523
$\mathbb{Z}/4\mathbb{Z}$	243	481
$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	312	726
$\mathbb{Z}/5\mathbb{Z}$	56	54
$\mathbb{Z}/6\mathbb{Z}$	183	208
$\mathbb{Z}/7\mathbb{Z}$	13	11
$\mathbb{Z}/8\mathbb{Z}$	21	16
$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$	51	60
$\mathbb{Z}/9\mathbb{Z}$	6	4
$\mathbb{Z}/10\mathbb{Z}$	12	8
$\mathbb{Z}/12\mathbb{Z}$	6	2
$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$	11	6
$\mathbb{Z}/15\mathbb{Z}$	1	0
$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$	2	1

Shafarevich-Tate Groups

Table: Ш

#Ш	#isom	first curve having #Ш	Norm($\mathfrak{n}$)
1	3191	$[1, \varphi + 1, \varphi, \varphi, 0]$	31
4	84	$[1, 1, 1, -110, -880]$	45
9	43	$[\varphi + 1, -\varphi, 1, -54686\varphi - 35336, -7490886\varphi - 4653177]$	76
16	16	$[1, \varphi, \varphi + 1, -4976733\varphi - 3075797, -6393196918\varphi - 3951212998]$	45
25	2	$[0, -1, 1, -7820, -263580]$	121
36	2	$[1, -\varphi + 1, \varphi, 1326667\varphi - 2146665, 880354255\varphi - 1424443332]$	1580

2: Finding all E attached to a newform g

The Modularity Conjecture

Modularity is critical to making systematic tables.

Conjecture

There is a bijection[a]

$$\{L(E,s) : E/F \text{ cond } \mathfrak{n}\} \xrightarrow{\text{conj} \cong} \{L(f,s) : \text{ newform } f \in S_{(2,2)}(\Gamma_0(\mathfrak{n}); \mathbb{Q})\}$$

[a] We consider L-series to be equal only if all of their Euler factors are equal!

Unpublished Remark (Taylor): If $E[3]|_{\mathrm{Gal}(\overline{\mathbb{Q}}/F(\zeta_3))}$ is absolutely irreducible, then modularity follows from recent work of Gee and Kisin.

Finding an E attached to a newform g

Theorem

Assume the modularity conjecture. There is an algorithm that takes as input a Hilbert modular newform $g \in S_{(2,2)}(\Gamma_0(\mathfrak{n}); \mathbb{Q})$ and outputs an elliptic curve E/F with $L(E,s) = L(g,s)$.

Proof.

By computing all the rational newforms in $S_{(2,2)}(\Gamma_0(\mathfrak{n}); \mathbb{Q})$, find a bound B so that the eigenvalues $a_{\mathfrak{p}}$ for $N(\mathfrak{p}) \leq B$ determine a newform. Enumerate the countably many elliptic curves E/F in any way you like; when you find one with conductor $\mathfrak{n}$, use the bound B to determine whether or not $L(E,s) = L(g,s)$. Since E corresponds to *some* newform, this procedure must terminate with the correct answer. $\square$

1. Similar argument for abelian varieties of GL_2-type.
2. Cremona: "this algorithm is not *respectable*!"

Finding an E attached to a newform g

1. **Naive enumeration** – previous slide
2. **Sieved enumeration** – use $a_{\mathfrak{p}}$ to impose congruence conditions
3. **Torsion families** – use $a_{\mathfrak{p}}$ to determine whether $\ell \mid \#E(F)$, and if so search over the family of curves with ℓ-torsion.
4. **Congruence families** – if you know E' and that $E'[\ell] \approx E[\ell]$, use Tom Fisher's explicit families.
5. **Twisting** – find a minimal conductor twist.
6. **Cremona-Lingham** – find curves with good reduction outside $\mathfrak{n}$.
7. **Dembele** – reverse engineer periods from special values of L-series.
8. **Elkies** – use the λ invariant.

Jon Bober's talk next will have a lot more to say about this.

Finding *all* E attached to a newform g

Compute the isogeny class of a curve using the following two steps repeatedly on each curve found until we find nothing new.

1. Use Billerey (2011) to compute a set S of possible prime degrees of isogenies $E \to E'$.
2. For each $\ell \in S$, use formulas (e.g., as in Kohel's thesis) to find all $\psi : E \to E'$ of degree ℓ.

Billerey in Code

```
def _plstar1(E, q):
    R.<x> = F[]
    t12 = 2048*x^12 -6144*x^10 + 6912*x^8 -3584*x^6 + 840*x^4 -72*x^2 + 1
    t12p = 2048*x^6 -6144*x^5 + 6912*x^4 -3584*x^3 + 840*x^2 -72*x + 1
    t24 = 2*(t12)^2 - 1
    #this is only for primes that have no ramification and have good reduction
    if len(F.primes_above(q)) == 1:
        w1 = 1 - 2*(q^12)*t12(x/(2*q)) + q^24
        t1 = E.change_ring(F.ideal(q).residue_field()).trace_of_frobenius()
        w = w1(t1)
        m = []
        for zee in factor(ZZ(w)):
            m.append(zee[0])
        return m
    else:
        v = F.primes_above(q)
        t1 = E.change_ring(v[0].residue_field()).trace_of_frobenius()
        t2 = E.change_ring(v[1].residue_field()).trace_of_frobenius()
        w1 = t12p(x^2/(4*q))
        w = 1 - 4*(q^12)*w1(t1)*w1(t2) - 2*(q^24)*(1- 2*(w1(t1)^2 + w1(t2)^2)) \
               - 4*(q^36)*w1(t1)*w1(t2) + q^48
        m = []
        for zee in factor(ZZ(w)):
            m.append(zee[0])
        return m

def _plstar12(E, q):
    #same caveat, only for unramified and good reduction
    if len(F.primes_above(q)) == 1:
       t1 = E.change_ring(F.prime_above(q).residue_field()).trace_of_frobenius()
       m = [q]
       try:
           for v in factor(t1):
```

```
            m.append(v[0])
        for v in factor(t1^2 - q^2):
            m.append(v[0])
        for v in factor(t1^2 - 4*q^2):
            m.append(v[0])
        for v in factor(t1^2 - 3*q^2):
            m.append(v[0])
        s1 = set(m)
        m = list(s1)
        return m
    except ArithmeticError:
        return 0
else:
    t1 = E.change_ring(F.primes_above(q)[0].residue_field()).trace_of_frobenius()
    t2 = E.change_ring(F.primes_above(q)[1].residue_field()).trace_of_frobenius()
    m = [q]
    try:
        for v in factor((t1^2 + t2^2 - q^2)^2 - 3*(t1^2)*(t2^2)):
            m.append(v[0])
        for v in factor(t1^2 - t2^2):
            m.append(v[0])
        for v in factor(t1^2 +t2^2 - 4*q^2):
            m.append(v[0])
        for v in factor((t1^2 + t2^2 - 3*q^2)^2 - (t1*t2)^2):
            m.append(v[0])
        s1 = set(m)
        m = list(s1)
        return m
    except ArithmeticError:
        return 0
```

```
def billerey_primes(E):
    ans = set([])
    Bad = [v[0] for v in E.conductor().norm().factor()]
    Pr = prime_range(1000)
    num = 0
    i = 0
    X = [set([3])]
    while num < 3:
        if not Pr[i] in Bad and Pr[i] != 5:
            try:
                X.append(set(_plstar1(E, Pr[i]) + _plstar12(E, Pr[i])))
                num += 1
            except TypeError:
                pass
        i += 1
    ans = (X[1].intersection(X[2])).intersection(X[3])
    ans = ans.union(set(Bad)).union(set([2,3,5]))
    return list(sorted(ans))
```

3: Finding newforms

Computing Hilbert Modular Forms over F

1. The algorithm is from Lassina Dembele's Ph.D. thesis. See his *Explicit computation of Hilbert modular forms on* $\mathbb{Q}(\sqrt{5})$ (2005).
2. Jacquet-Langlands: Computing Hecke module of Hilbert modular forms of level $\mathfrak{n}$ over F same as computing Hecke module with basis that right ideal classes in a certain order (of level $\mathfrak{n}$) in the Hamilton quaternion algebra over F.
3. Dembele: Computing right ideal classes same as computing $\mathbb{P}^1(R/\mathfrak{n})$, where $R = \mathbb{Z}[\varphi] \subset F$.

Dembele's Algorithm in One Slide

1. Hamiltonian quaternions $F[i,j,k]$ ramified at the infinite places.
2. Maximal order
$$S = R\Big[\frac{1}{2}(1-\overline{\varphi}i+\varphi j),\ \frac{1}{2}(-\overline{\varphi}i+j+\varphi k),\ \frac{1}{2}(\varphi i-\overline{\varphi}j+k),\ \frac{1}{2}(i+\varphi j-\overline{\varphi}k)\Big].$$
3. $\mathbb{P}^1(R/\mathfrak{n})$ = equivalence classes of column vectors with two coprime entries $a, b \in R/\mathfrak{n}$ modulo the action of $(R/\mathfrak{n})^*$.
4. For each $\mathfrak{p} \mid \mathfrak{n}$, fix *choice* of isomorphism $F[i,j,k] \otimes F_\mathfrak{p} \approx M_2(F_\mathfrak{p})$, which induces a *choice* of left action of S^* on $\mathbb{P}^1(R/\mathfrak{n})$.
5. Jacquet-Langlands: There's an isomorphism of $\mathbb{T}$-modules
$$\mathbb{C}[S^*\backslash\mathbb{P}^1(R/\mathfrak{n})] \cong M_{(2,2)}(\Gamma_0(\mathfrak{n})).$$
6. S^* acts through the *octonian* group (which is finite and explicit).
7. $T_\mathfrak{p}([x]) = \sum[\alpha x]$, where sum is over the classes $[\alpha] \in S/S^*$ with $N_{\mathrm{red}}(\alpha) = \pi_\mathfrak{p}$, where $\pi_\mathfrak{p}$ is fixed choice of positive generator of $\mathfrak{p}$.

Implementation Notes

1. Critical that we can compute with $\mathbb{P}^1(R/\mathfrak{n})$ very, very, very quickly.
2. Prime power $\mathfrak{n} = \mathfrak{p}^e$ case: Each $[x : y] \in \mathbb{P}^1(R/\mathfrak{p}^e)$ has a unique representative $[1 : b]$ or $[a : 1]$ with a divisible by $\mathfrak{p}$. Easy to put any $[x : y]$ in this canonical form.
3. General case: factor $\mathfrak{n} = \prod_{i=1}^m \mathfrak{p}_i^{e_i}$. Have a bijection $\mathbb{P}^1(R/\mathfrak{n}) \cong \prod_{i=1}^m \mathbb{P}^1(R/\mathfrak{p}_i^{e_i})$, thus reducing to the prime power case. Represent elements of $R/\mathfrak{n}$ as m-tuples in $\prod R/\mathfrak{p}_i^{e_i}$, making computation of the bijection trivial.
4. (Drew Sutherland-style tricks) We minimize dynamic memory allocation speeding up the code by an order of magnitude, by making some arbitrary bounds.
5. Painful to implement, but it is *fast*. Not included in Sage yet: `http://trac.sagemath.org/sage_trac/ticket/12465`

What Next?

My group's project at the 2012 MRC in Snowbird Utah (June 24–30, 2012) will be to compute Hilbert newforms in $S_{(2,2)}(\Gamma_0(\mathfrak{n}))$ as far as possible, gather *arithmetic statistics* about them (e.g., analytic ranks), make conjectures, and perhaps prove something.

Example Goal: Does the first elliptic curve of rank 3 have norm conductor 163^2 or not?

rank	norm($\mathfrak{n}$)	equation	person
0	31 (prime)	$[1, \varphi+1, \varphi, \varphi, 0]$	Dembele
1	199 (prime)	$[0, -\varphi-1, 1, \varphi, 0]$	Dembele
2	1831 (prime)	$[0, -\varphi, 1, -\varphi-1, 2\varphi+1]$	Dembele
3	$26{,}569 = 163^2$	$[0, 0, 1, -2, 1]$	Elkies
4	1,209,079 (prime)	$[1, -1, 0, -8-12\varphi, 19+30\varphi]$	Elkies
5	64,004,329	$[0, -1, 1, -9-2\varphi, 15+4\varphi]$	Elkies

Epilogue (or Prologue)

```
On Wed, Feb 2, 2011 at 12:18 PM, William Stein <wstein@gmail.com> wrote:
> Hi John [Voight],
>
> I'm planning to try to say something about these sorts of things...
> mainly that I'm ignorant in each case.  But I'm curious what thoughts
> you might have about these...
>
>       -- stein-watkins style search
>       -- elkies approach: Q(sqrt(5)) curves
>       -- rank info
>       -- gens (simon 2-descent output)
>       -- L-function (fast computation of a_p?)
>       -- congruence number
>       -- isogeny class (enumerate)
>       -- root number
>       -- torsion subgroup
>       -- tamagawa numbers
>       -- all integral points
>       -- Kodaira symbols
>       -- zeros of L(E/F,s) in critical strip
>       -- notion of "canonical" minimal weierstrass model
>       -- picture
>       -- height pairing / regulator
>       -- heegner points
>       -- #Sha(E/F) -- when hypo of Zhang's work satisfied, there is hope.
>       -- images of Galois reps?
>       -- as much as possible of the above for modular abelian varieties A_f.
```