

# Generating the Hecke algebra

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## Abstract

Let  $\mathbb{T}$  be the Hecke algebra associate to weight  $k$  modular forms for  $X_0(N)$ . We give a bound for the number of Hecke operators  $T_n$  needed to generate  $\mathbb{T}$  as a  $\mathbb{Z}$ -module.

## Introduction

In this note we apply a theorem of Sturm [S] to prove a bound on the number of Hecke operators needed to generate the Hecke algebra as a  $\mathbb{Z}$ -module. This bound was observed by Ken Ribet, but has not been written down. In section 2 we record our notation and some standard theorems. In section 3 we state Sturm's theorem and use it to deduce a bound on the number of generators of the Hecke algebra.

## 1 Modular forms and Hecke operators

Let  $N$  and  $k$  be positive integers and let  $M_k(N) = M_k(\Gamma_0(N))$  be the  $\mathbb{C}$ -vector space of weight  $k$  modular forms on  $X_0(N)$ . This space can be viewed as the set of functions  $f(z)$ , holomorphic on the upper half-plane, such that

$$f(z) = f|[\gamma]_k(z) := (cz + d)^{-k} f\left(\frac{az + b}{cz + d}\right)$$

for all  $\gamma \in \Gamma_0(N)$ , and such that  $f$  satisfies a certain holomorphic condition at the cusps.

Any  $f \in M_k(N)$  has a Fourier expansion

$$f = a_0(f) + a_1(f)q + a_2(f)q^2 + \cdots = \sum a_n q^n \in \mathbb{C}[[q]]$$

where  $q = e^{2\pi iz}$ . The map sending  $f$  to its  $q$ -expansion is an injective map  $M_k(N) \hookrightarrow \mathbb{C}[[q]]$  called the  $q$ -expansion map. Define  $M_k(N; \mathbb{Z})$  to be the inverse

image of  $\mathbb{Z}[[q]]$  under this map. It is known (see §12.3, [DI]) that

$$M_k(N) = M_k(N; \mathbb{Z}) \otimes \mathbb{C}.$$

For any ring  $R$ , define  $M_k(N; R) := M_k(N; \mathbb{Z}) \otimes_{\mathbb{Z}} R$ .

Let  $p$  be a prime. Define two operators on  $\mathbb{C}[[q]]$ :

$$V_p\left(\sum a_n q^n\right) = \sum a_n q^{np}$$

and

$$U_p\left(\sum a_n q^n\right) = \sum a_{np} q^n.$$

The Hecke operator  $T_p$  acts on  $q$ -expansions by

$$T_p = U_p + \varepsilon(p)p^{k-1}V_p$$

where  $\varepsilon(p) = 1$ , unless  $p|N$  in which case  $\varepsilon(p) = 0$ . If  $m$  and  $n$  are coprime, the Hecke operators satisfy  $T_{nm} = T_n T_m = T_m T_n$ . If  $p$  is a prime and  $r \geq 2$ ,

$$T_{p^r} = T_{p^{r-1}} T_p - \varepsilon(p)p^{k-1}T_{p^{r-2}}.$$

The  $T_n$  are linear maps which preserves  $M_k(N; \mathbb{Z})$ . The Hecke algebra  $\mathbb{T} = \mathbb{T}(N) = \mathbb{Z}[T_1, T_2, T_3, \dots]$ , which is viewed as a subring of the ring of linear endomorphisms of  $M_k(N)$ , is a finite commutative  $\mathbb{Z}$ -algebra.

**Proposition 1.1.** *Let  $\sum a_n q^n$  be the  $q$ -expansion of  $f \in M_k(N)$  and let  $\sum b_n q^n$  be the  $q$ -expansion of  $T_m f$ . Then the coefficients  $b_n$  are given by*

$$b_n = \sum_{d|(m,n)} \varepsilon(d)d^{k-1}a_{mn/d^2}.$$

Note in particular that  $a_1(T_m f) = a_m(f)$ .

*Proof.* Proposition 3.4.3, [DI]. □

**Proposition 1.2.** *For any ring  $R$ , there is a perfect pairing*

$$\mathbb{T}_R \otimes_R M_k(N; R) \rightarrow R, \quad (T, f) \mapsto a_1(Tf),$$

where  $\mathbb{T}_R = \mathbb{T} \otimes_{\mathbb{Z}} R$ .

*Proof.* Proposition 12.4.13, [DI]. □

## 2 Bounding the number of generators

Let  $\mu(N) = N \prod_{p|N} (1 + \frac{1}{p})$  be the index of  $\Gamma_0(N)$  in  $\mathrm{SL}_2(\mathbb{Z})$ .

**Theorem 2.1.** *Let  $\lambda$  be a prime ideal in the ring of integers  $\mathcal{O}$  of some number field. Suppose  $f \in M_k(N; \mathcal{O})$  is such that  $a_n(f) \equiv 0 \pmod{\lambda}$  for  $n \leq \frac{k}{12}\mu(N)$ . Then  $f \equiv 0 \pmod{\lambda}$ .*

*Proof.* Theorem 1, [S]. □

Denote by  $\lceil x \rceil$  the smallest integer  $\geq x$ .

**Proposition 2.2.** *Suppose  $f \in M_k(N)$  and*

$$a_n(f) = 0 \quad \text{for } n \leq r = \left\lceil \frac{k}{12} \mu(N) \right\rceil.$$

*Then  $f = 0$ .*

*Proof.* We must show that the composite map

$$M_k(N) \hookrightarrow \mathbb{C}[[q]] \rightarrow \mathbb{C}[[q]]/(q^{r+1})$$

is injective. Because  $\mathbb{C}$  is a flat  $\mathbb{Z}$ -module, it suffices to show that the map  $\Phi : M_k(N; \mathbb{Z}) \rightarrow \mathbb{Z}[[q]]/(q^{r+1})$  is injective. Suppose  $\Phi(f) = 0$ , and let  $p$  be a prime number. Then  $a_n(f) = 0$  for  $n \leq r$ , hence plainly  $a_n(f) \equiv 0 \pmod{p}$  for any such  $n$ . By Theorem 2.1, it follows that  $f \equiv 0 \pmod{p}$ . Repeating this argument shows that the coefficients of  $f$  are divisible by all primes  $p$ , i.e., they are 0. □

**Theorem 2.3.** *The Hecke algebra is generated as a  $\mathbb{Z}$ -module by  $T_1, \dots, T_r$  where  $r = \lceil \frac{k}{12} \mu(N) \rceil$ .*

*Proof.* Let  $A$  be the submodule of  $\mathbb{T}$  generated by  $T_1, T_2, \dots, T_r$ . Consider the exact sequence of additive abelian groups

$$0 \rightarrow A \xrightarrow{i} \mathbb{T} \rightarrow \mathbb{T}/A \rightarrow 0.$$

Let  $p$  be a prime and tensor with  $\mathbb{F}_p$  to obtain

$$A \otimes \mathbb{F}_p \xrightarrow{\tilde{i}} \mathbb{T} \otimes \mathbb{F}_p \rightarrow (\mathbb{T}/A) \otimes \mathbb{F}_p \rightarrow 0$$

(tensor product is right exact). Put  $R = \mathbb{F}_p$  in Proposition 1.2, and suppose  $f \in M_k(N, \mathbb{F}_p)$  pairs to 0 with each of  $T_1, \dots, T_r$ . Then by Proposition 1.1,  $a_m(f) = a_1(T_m f) = 0$  in  $\mathbb{F}_p$  for each  $m$ ,  $1 \leq m \leq r$ . By Theorem 2.1 it follows that  $f = 0$ . Thus the pairing, when restricted to the image of  $A \otimes \mathbb{F}_p$  in  $\mathbb{T} \otimes \mathbb{F}_p$ , is also perfect and so

$$\dim_{\mathbb{F}_p} \tilde{i}(A \otimes \mathbb{F}_p) = \dim_{\mathbb{F}_p} M_k(N, \mathbb{F}_p) = \dim_{\mathbb{F}_p} \mathbb{T} \otimes \mathbb{F}_p.$$

We see that  $(\mathbb{T}/A) \otimes \mathbb{F}_p = 0$ ; repeating the argument for all  $p$  shows that the finitely generated abelian group  $\mathbb{T}/A$  must be trivial. □

## References

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