Questions about 144169

B. Mazur

April 6, 2006

Let $p = 144169$ which, despite appearances, is a prime number. Let $\Lambda := \mathbf{Z}_p[[\Gamma]]$ where $\Gamma \subset \mathbf{Z}_p^*$ is the group of 1-units in $\mathbf{Z}_p^*$; as usual choose a topological generator $\gamma \in \Gamma$, but don't bank too much on it, writing $z := \gamma - 1$ so that Λ is equal to the power series ring in the one variable z over $\mathbf{Z}_p$. Let $\mathbf{T}$ denote the component of the Hida-Hecke algebra for this prime number p, and tame level $N = 1$, with "weight mod $p-1$" equal to 24. The computations of William Stein (see his preprint *The two weight* 24 *cuspforms are ordinary*) plus standard theorems tell us that $\mathbf{T}$ is irreducible and a quadratic finite flat algebra over Λ. The discriminant ideal of the Λ-algebra $\mathbf{T}$ is of course a principal ideal, and has a generator of the form $\Delta(z) = p^\mu \cdot f(z) \in \mathbf{Z}_p[z] \subset \Lambda$ where $f(z)$ is a distinguished polynomial (i.e., a monic poynomial such that all coefficients but the top one lie in $p\mathbf{Z}_p$). If

$$\mathbf{T}^{\text{weight }24} := \mathbf{T} \otimes_\Lambda \mathbf{Z}_p^{\text{weight }24}$$

where $\mathbf{Z}_p^{\text{weight }24}$ is the quotient of Λ by the ideal corresponding to passage to weight 24, then we have that

$$\mathbf{T}^{\text{weight }24} = \mathbf{Z}_p[\sqrt{p}],$$

the two cuspforms of level 1 and weight 24 having q-expansions given by the conjugates of

$$q + (540 - 12\sqrt{1}44169)q^2 + (169740 + 576\sqrt{1}44169)q^3 + \ldots$$

so we have that

$$\mathrm{ord}_p(\Delta(z_{24}) = 1,$$

where, without bothering to normalize, let $z_{24} \in \mathbf{Z}_p$ be whatever one it has to be to get the value after specializing to weight 24.

Following the conventions of Iwasawa theory, put $\lambda :=$ the degree of $f(z)$ and the question is: what is μ and what is λ, where the above analysis gives us that either

- $\mu = 1$, $\lambda = 0$, and the rigid-analytic Hida family associated to $\mathbf{T}$ is unramified over weight space, or
- $\mu = 0$, $\lambda > 0$, and the rigid-analytic Hida family associated to $\mathbf{T}$ is ramified over weight space.

I'm guessing that the latter possibility happens, following a more general conjecture that I will begin to describe.

Let ℓ a prime number, k an integer modulo $\ell - 1$, and tame level $N = 1$. Put $\Lambda_\ell := \mathbf{Z}_\ell[[\Gamma_\ell]]$ where, Γ_ℓ is the group of 1-units in $\mathbf{Z}_\ell^*$ as before. Let $\mathbf{T}_{\ell,k}$ be the Hida-Hecke Λ_ℓ-algebra whose characters give the Hecke eigenvalues of ordinary ℓ-adic modular eigenforms of tame level $N = 1$, and "weight mod ℓ" equal to k. Decompose the complete, noetherian, semi-local ring $\mathbf{T}_{\ell,k}$ as a product of finitely many local Λ_ℓ-algebras:

$$\mathbf{T}_{\ell,k} = \prod_\nu \mathbf{T}_{\ell,k,\nu}.$$

Let $\Delta_{\ell,k,\nu}$ be the discriminant, as before, of the Λ_ℓ-algebra $\mathbf{T}_{\ell,k,\nu}$. Let $\mu_{\ell,k,\nu}$ and $\lambda_{\ell,k,\nu}$ be the "$\mu-$ and $\lambda-$invariants" of $\Delta_{\ell,k,\nu}$. Associated (in the usual way) to each of the local rings $\mathbf{T}_{\ell,k,\nu}$ we have a canonical semi-simple residual representation,

$$\bar{\rho}_{\ell,k,\nu} : G_{\mathbf{Q},\{\ell,\infty\}} \to \mathrm{GL}_2(\mathbf{F}_{\ell,k,\nu}),$$

where $\mathbf{F}_{\ell,k,\nu}$ is the residue field of the local ring $\mathbf{T}_{\ell,k,\nu}$.

Denote by

$$\bar{\rho}^{(2)}_{\ell,k,\nu} : G_{\mathbf{Q},\{\ell,\infty\}} \to \mathrm{GL}_3(\mathbf{F}_{\ell,k,\nu})$$

the symmetric square of the residual representation $\bar{\rho}_{\ell,k,\nu}$.

Following in the spirit of conjectures of Greenberg and Greenberg-Vatsal, one might hope for

Conjecture. If the (residual) symmetric square representation $\bar{\rho}^{(2)}_{\ell,k,\nu}$ is irreducible, then $\mu_{\ell,k,\nu} = 0$.

Since the (residual) symmetric square representation associated to the Hida-Hecke ring $\mathbf{T}$ that we began our discussion with (i.e., $\ell = p(= 144169)$ and $k = 24$, and $\mathbf{T} = \mathbf{T}_{144169,24}$) is indeed irreducible, we are guessing that its corresponding μ-invariant vanishes, as discussed above.

Equally interesting here would be cases where the (residual) symmetric square representation is *reducible* for we have specific conjectures and results regarding μ-invariants (not for symmetric squares of residual representations arising from Hida-Hecke algebras, but rather) for Selmer modules attached to the indecomposable, but not irreducible, residual two-dimensional representations that are suggestive here; I am referring to results of Greenberg-Vatsal, and of Mak Trifkovic.

While we are at it, we might ask with there is lower bound for, of all things, ℓ times the ord_ℓ's of the zeroes of $\Delta_{\ell,k,\nu}$, at least when $\bar{\rho}^{(2)}_{\ell,k,\nu}$ is irreducible.