6.7 Applications of Taylor Series

Final exam: Wednesday, March 22, 7-10pm in PCYNH 109. Bring ID!
Last Quiz 4: Today (last one)
Today: 11.12 Applications of Taylor Polynomials
Next; Differential Equations

This section is about an example in the theory of relativity. Let m be the (relativistic) mass of an object and m_0 be the mass at rest (rest mass) of the object. Let v be the velocity of the object relative to the observer, and let c be the speed of light. These three quantities are related as follows:

$$m = \frac{m_0}{\sqrt{1-\dfrac{v^2}{c^2}}} \qquad \text{(relativistic) mass}$$

The total energy of the object is mc^2:

$$\boxed{E = mc^2.}$$

In relativity we define the kinetic energy to be

$$K = mc^2 - m_0c^2. \tag{6.7.1}$$

What? Isn't the kinetic energy $\frac{1}{2}m_0v^2$?

Notice that

$$mc^2 - m_0c^2 = \frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}} - m_0c^2 = m_0c^2\left[\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} - 1\right].$$

Let

$$f(x) = (1-x)^{-\frac{1}{2}} - 1$$

Let's compute the Taylor series of f. We have

$$\begin{aligned}
f(x) &= (1-x)^{-\frac{1}{2}} - 1\\
f'(x) &= \frac{1}{2}(1-x)^{-\frac{3}{2}}\\
f''(x) &= \frac{1}{2}\cdot\frac{3}{2}(1-x)^{-\frac{5}{2}}\\
f^{(n)}(x) &= \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n}(1-x)^{-\frac{2n+1}{2}}.
\end{aligned}$$

Thus

$$f^{(n)}(0) = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n}.$$

Hence

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ &= \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n \cdot n!} x^n \\ &= \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \frac{35}{128}x^4 + \cdots \end{aligned}$$

We now use this to analyze the kinetic energy (6.7.1):

$$\begin{aligned} mc^2 - m_0c^2 &= m_0c^2 \cdot f\left(\frac{v^2}{c^2}\right) \\ &= m_0c^2 \cdot \left(\frac{1}{2} \cdot \frac{v^2}{c^2} + \frac{3}{8} \cdot \frac{v^2}{c^2} + \cdots\right) \\ &= \frac{1}{2}m_0v^2 + m_0c^2 \cdot \left(\frac{3}{8}\frac{v^2}{c^2} + \cdots\right) \end{aligned}$$

And we can ignore the higher order terms if $\frac{v^2}{c^2}$ is small. But how small is "small" enough, given that $\frac{v^2}{c^2}$ appears in an infinite sum?

6.7.1 Estimation of Taylor Series

Suppose

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Write

$$R_N(x) := f(x) - \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

We call

$$T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

the Nth degree *Taylor polynomial*. Notice that

$$\lim_{N\to\infty} T_N(x) = f(x)$$

if and only if

$$\lim_{N\to\infty} R_N(x) = 0.$$

We would like to estimate $f(x)$ with $T_N(x)$. We need an estimate for $R_N(x)$.

Theorem 6.7.1 (Taylor's theorem). *If $|f^{(N+1)}(x)| \le M$ for $|x-a| \le d$, then*

$$|R_N(x)| \le \frac{M}{(N+1)!}|x-a|^{N+1} \qquad \text{for } |x-a| \le d.$$

For example, if $N = 0$, this says that

$$|R(x)| = |f(x) - f(a)| \leq M|x - a|,$$

i.e.,

$$\left| \frac{f(x) - f(a)}{x - a} \right| \leq M,$$

which should look familiar from a previous class (Mean Value Theorem).

Applications:

1. One can use Theorem 6.7.1 to prove that functions converge to their Taylor series.

2. Returning to the relativity example above, we apply Taylor's theorem with $N = 1$ and $a = 0$. With $x = -v^2/c^2$ and M any number such that $|f''(x)| \leq M$, we have

   $$|R_1(x)| \leq \frac{M}{2}x^2.$$

   For example, if we assume that $|v| \leq 100m/s$ we use

   $$|f''(x)| \leq \frac{3}{2}(1 - 100^2/c^2)^{-5/2} = M.$$

   Using $c = 3 \times 10^8 m/s$, we get

   $$|R_1(x)| \leq 4.17 \cdot 10^{-10} \cdot m_0.$$

   Thus for $v \leq 100m/s \sim 225$mph, then the error in throwing away relativistic factors is $10^{-10}M$. This is like 200 feet out of the distance to the sun (93 million miles). So relativistic and Newtonian kinetic energies are almost the same for reasonable speeds.