6.6 Taylor Series

Final exam: Wednesday, March 22, 7-10pm in PCYNH 109. Bring ID!
Last Quiz 4: This Friday
Next: 11.10 Taylor and Maclaurin series
Next: 11.12 Applications of Taylor Polynomials
Midterm Letters:
A, 32–38
B, 26–31
C, 20–25
D, 14–19
Mean: 23.4, Standard Deviation: 7.8, High: 38, Low: 6.

Example 6.6.1. Suppose we have a degree-3 (cubic) polynomial p and we know that $p(0) = 4$, $p'(0) = 3$, $p''(0) = 4$, and $p'''(0) = 6$. Can we determine p? Answer: Yes! We have

$$\begin{aligned} p(x) &= a + bx + cx^2 + dx^3 \\ p'(x) &= b + 2cx + 3dx^2 \\ p''(x) &= 2c + 6dx \\ p'''(x) &= 6d \end{aligned}$$

From what we mentioned above, we have:

$$\begin{aligned} a &= p(0) = 4 \\ b &= p'(0) = 3 \\ c &= \frac{p''(0)}{2} = 2 \\ d &= \frac{p'''(0)}{6} = 1 \end{aligned}$$

Thus

$$p(x) = 4 + 3x + 2x^2 + x^3.$$

Amazingly, we can use the idea of Example 6.6.1 to compute power series expansions of functions. E.g., we will show below that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Convergent series are determined by the values of their derivatives.

Consider a general power series

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

We have

$$\begin{aligned} c_0 &= f(a) \\ c_1 &= f'(a) \\ c_2 &= \frac{f''(a)}{2} \\ \cdots \\ c_n &= \frac{f^{(n)}(a)}{n!}, \end{aligned}$$

where for the last equality we use that

$$f^{(n)}(x) = n!c_n + (x-a)(\cdots + \cdots)$$

Remark 6.6.2. The definition of 0! is 1 (it's the empty product). The empty sum is 0 and the empty product is 1.

Theorem 6.6.3 (Taylor Series). *If $f(x)$ is a function that equals a power series centered about a, then that power series expansion is*

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots \end{aligned}$$

Remark 6.6.4. WARNING: There are functions that have all derivatives defined, but do not equal their Taylor expansion. E.g., $f(x) = e^{-1/x^2}$ for $x \neq 0$ and $f(0) = 0$. It's Taylor expansion is the 0 series (which converges everywhere), but it is not the 0 function.

Definition 6.6.5 (Maclaurin Series). A *Maclaurin series* is just a Taylor series with $a = 0$. I will not use the term "Maclaurin series" ever again (it's common in textbooks).

Example 6.6.6. Find the Taylor series for $f(x) = e^x$ about $a = 0$. We have $f^{(n)}(x) = e^x$. Thus $f^{(n)}(0) = 1$ for all n. Hence

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

What is the radius of convergence? Use the ratio test:

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)!}x^{n+1}}{\frac{1}{n!}x^n} \right| &= \lim_{n\to\infty} \frac{n!}{(n+1)!}|x| \\ &= \lim_{n\to\infty} \frac{|x|}{n+1} = 0, \qquad \text{for any fixed } x. \end{aligned}$$

Thus the radius of convergence is ∞.

Example 6.6.7. Find the Taylor series of $f(x) = \sin(x)$ about $x = \frac{\pi}{2}$.[1] We have

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}\left(\frac{\pi}{2}\right)}{n!}\left(x - \frac{\pi}{2}\right)^n.$$

[1] Evidently this expansion was first found in India by Madhava of Sangamagrama (1350-1425).

To do this we have to *puzzle out a pattern*:

$$\begin{aligned}
f(x) &= \sin(x)\\
f'(x) &= \cos(x)\\
f''(x) &= -\sin(x)\\
f'''(x) &= -\cos(x)\\
f^{(4)}(x) &= \sin(x)
\end{aligned}$$

First notice how the signs behave. For $n = 2m$ even,

$$f^{(n)}(x) = f^{(2m)}(x) = (-1)^{n/2}\sin(x)$$

and for $n = 2m+1$ odd,

$$f^{(n)}(x) = f^{(2m+1)}(x) = (-1)^m\cos(x) = (-1)^{(n-1)/2}\cos(x)$$

For $n = 2m$ even we have

$$f^{(n)}(\pi/2) = f^{(2m)}\left(\frac{\pi}{2}\right) = (-1)^m.$$

and for $n = 2m+1$ odd we have

$$f^{(n)}(\pi/2) = f^{(2m+1)}\left(\frac{\pi}{2}\right) = (-1)^m\cos(\pi/2) = 0.$$

Finally,

$$\begin{aligned}
\sin(x) &= \sum_{n=0}^{\infty}\frac{f^{(n)}(\pi/2)}{n!}(x-\pi/2)^n\\
&= \sum_{m=0}^{\infty}\frac{(-1)^m}{(2m)!}\left(x-\frac{\pi}{2}\right)^{2m}.
\end{aligned}$$

Next we use the ratio test to compute the radius of convergence. We have

$$\begin{aligned}
\lim_{m\to\infty}\frac{\left|\dfrac{(-1)^{m+1}}{(2(m+1))!}\left(x-\dfrac{\pi}{2}\right)^{2(m+1)}\right|}{\left|\dfrac{(-1)^m}{(2m)!}\left(x-\dfrac{\pi}{2}\right)^{2m}\right|} &= \lim_{m\to\infty}\frac{(2m)!}{(2m+2)!}\left(x-\frac{\pi}{2}\right)^2\\
&= \lim_{m\to\infty}\frac{\left(x-\frac{\pi}{2}\right)^2}{(2m+2)(2m+1)}
\end{aligned}$$

which converges for each x. Hence $R = \infty$.

Example 6.6.8. Find the Taylor series for $\cos(x)$ about $a = 0$. We have $\cos(x) = \sin\left(x+\frac{\pi}{2}\right)$. Thus from Example 6.6.7 (with infinite radius of convergence) and that the Taylor expansion is unique, we have

$$\begin{aligned}
\cos(x) &= \sin\left(x+\frac{\pi}{2}\right)\\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(x+\frac{\pi}{2}-\frac{\pi}{2}\right)^{2n}\\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}
\end{aligned}$$