6.4 Tests for Convergence

Final exam: Wednesday, March 22, 7-10pm in PCYNH 109.
Quiz 4: Next Friday
Today: 11.6: Ratio and Root tests
Next: 11.8 Power Series
11.9 Functions defined by power series

6.4.1 The Comparison Test

Theorem 6.4.1 (The Comparison Test). *Suppose* $\sum a_n$ *and* $\sum b_n$ *are series with all* a_n *and* b_n *positive and* $a_n \leq b_n$ *for each* n.

1. *If* $\sum b_n$ *converges, then so does* $\sum a_n$.
2. *If* $\sum a_n$ *diverges, then so does* $\sum b_n$.

Proof Sketch. The condition of the theorem implies that for any k,

$$\sum_{n=1}^{k} a_n \leq \sum_{n=1}^{k} b_n,$$

from which each claim follows. □

Example 6.4.2. Consider the series $\sum_{n=1}^{\infty} \frac{7}{3n^2+2n}$. For each n we have

$$\frac{7}{3n^2+2n} \leq \frac{7}{3} \cdot \frac{1}{n^2}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, Theorem 6.4.1 implies that $\sum_{n=1}^{\infty} \frac{7}{3n^2+2n}$ also converges.

Example 6.4.3. Consider the series $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$. It diverges since for each $n \geq 3$ we have

$$\frac{\ln(n)}{n} \geq \frac{1}{n},$$

and $\sum_{n=3}^{\infty} \frac{1}{n}$ diverges.

6.4.2 Absolute and Conditional Convergence

Definition 6.4.4 (Converges Absolutely). We say that $\sum_{n=1}^{\infty} a_n$ *converges absolutely* if $\sum_{n=1}^{\infty} |a_n|$ converges.

For example,

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$$

converges, but does *not* converge absolutely (it converges “conditionally”, though we will not explain why in this book).

6.4.3 The Ratio Test

Recall that $\sum_{n=1}^{\infty} a_n$ is a geometric series if and only if $a_n = ar^{n-1}$ for some fixed a and r. Here we call r the *common ratio.* Notice that the ratio of any two successive terms is r:

$$\frac{a_{n+1}}{a_n} = \frac{ar^n}{ar^{n-1}} = r.$$

Moreover, we have $\sum_{n=1}^{\infty} ar^{n-1}$ converges (to $\frac{a}{1-r}$) if and only if $|r| < 1$ (and, of course it diverges if $|r| \geq 1$).

Example 6.4.5. For example, $\sum_{n=1}^{\infty} 3\left(\frac{2}{3}\right)^{n-1}$ converges to $\frac{3}{1-\frac{2}{3}} = 9$. However, $\sum_{n=1}^{\infty} 3\left(\frac{3}{2}\right)^{n-1}$ diverges.

Theorem 6.4.6 (Ratio Test). *Consider a sum $\sum_{n=1}^{\infty} a_n$. Then*

1. *If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$ then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.*
2. *If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges.*
3. *If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L = 1$ then we may conclude nothing from this!*

Proof. We will only prove 1. Assume that we have $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$. Let $r = \frac{L+1}{2}$, and notice that $L < r < 1$ (since $0 \leq L < 1$, so $1 \leq L+1 < 2$, so $1/2 \leq r < 1$, and also $r - L = (L+1)/2 - L = (1-L)/2 > 0$).

Since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$, there is an N such that for all $n > N$ we have

$$\left|\frac{a_{n+1}}{a_n}\right| < r, \qquad \text{so} \qquad |a_{n+1}| < |a_n| \cdot r.$$

Then we have

$$\sum_{n=N+1}^{\infty} |a_n| < |a_{N+1}| \cdot \sum_{n=0}^{\infty} r^n.$$

Here the common ratio for the second one is $r < 1$, hence thus the right-hand series converges, so the left-hand series converges. □

Example 6.4.7. Consider $\displaystyle\sum_{n=1}^{\infty} \frac{(-10)^n}{n!}$. The ratio of successive terms is

$$\left|\frac{\frac{(-10)^{n+1}}{(n+1)!}}{\frac{(-10)^n}{n!}}\right| = \frac{10^{n+1}}{(n+1)n!} \cdot \frac{n!}{10^n} = \frac{10}{n+1} \to 0 < 1.$$

Thus this series converges *absolutely.* Note, the minus sign is missing above since in the ratio test we take the limit of the absolute values.

Example 6.4.8. Consider $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{3^{1+3n}}$. We have

$$\left|\frac{\frac{(n+1)^{n+1}}{3\cdot(27)^{n+1}}}{\frac{n^n}{3^{1+3n}}}\right| = \frac{(n+1)(n+1)^n}{27\cdot 27^n} \cdot \frac{27^n}{n^n} = \frac{n+1}{27} \cdot \left(\frac{n+1}{n}\right)^n \to +\infty$$

Thus our series diverges. (Note here that we use that $\left(\frac{n+1}{n}\right)^n \to e$.)

Example 6.4.9. Let's apply the ratio test to $\sum_{n=1}^{\infty} \frac{1}{n}$. We have

$$\lim_{n\to\infty} \left| \frac{\frac{1}{n+1}}{\frac{1}{n}} \right| = \frac{1}{n+1} \cdot \frac{n}{1} = \frac{n}{n+1} \to 1.$$

This tells us nothing. If this happens... do something else! E.g., in this case, use the integral test.

6.4.4 The Root Test

Since e and ln are inverses, we have $x = e^{\ln(x)}$. This implies the very useful fact that

$$x^a = e^{\ln(x^a)} = e^{a\ln(x)}.$$

As a sample application, notice that for any nonzero c,

$$\lim_{n\to\infty} c^{\frac{1}{n}} = \lim_{n\to\infty} e^{\frac{1}{n}\log(c)} = e^0 = 1.$$

Similarly,

$$\lim_{n\to\infty} n^{\frac{1}{n}} = \lim_{n\to\infty} e^{\frac{1}{n}\log(n)} = e^0 = 1,$$

where we've used that $\lim_{n\to\infty} \frac{\log(n)}{n} = 0$, which we could prove using L'Hopital's rule.

Theorem 6.4.10 (Root Test). *Consider the sum* $\sum_{n=1}^{\infty} a_n$*.*

1. *If* $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = L < 1$*, then* $\sum_{n=1}^{\infty} a_n$ *convergest absolutely.*
2. *If* $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = L > 1$*, then* $\sum_{n=1}^{\infty} a_n$ *diverges.*
3. *If* $L = 1$*, then we may conclude nothing from this!*

Proof. We apply the comparison test (Theorem 6.4.1). First suppose $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = L < 1$. Then there is a N such that for $n \geq N$ we have $|a_n|^{\frac{1}{n}} < k < 1$. Thus for such n we have $|a_n| < k^n < 1$. The geometric series $\sum_{i=N}^{\infty} k^i$ converges, so $\sum_{i=N}^{\infty} |a_n|$ also does, by Theorem 6.4.1. If $|a_n|^{\frac{1}{n}} > 1$ for $n \geq N$, then we see that $\sum_{i=N}^{\infty} |a_n|$ diverges by comparing with $\sum_{i=N}^{\infty} 1$. □

Example 6.4.11. Let's apply the root test to

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{r} \sum_{n=1}^{\infty} r^n.$$

We have

$$\lim_{n\to\infty} |r^n|^{\frac{1}{n}} = |r|.$$

Thus the root test tells us exactly what we already know about convergence of the geometry series (except when $|r| = 1$).

Example 6.4.12. The sum $\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n$ is a candidate for the root test. We have

$$\lim_{n\to\infty} \left|\left(\frac{n^2+1}{2n^2+1}\right)^n\right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2+1}{2n^2+1} = \lim_{n\to\infty} \frac{1+\frac{1}{n^2}}{2+\frac{1}{n^2}} = \frac{1}{2}.$$

Thus the series converges.

Example 6.4.13. The sum $\sum_{n=1}^{\infty} \left(\frac{2n^2+1}{n^2+1}\right)^n$ is a candidate for the root test. We have

$$\lim_{n\to\infty} \left|\left(\frac{2n^2+1}{n^2+1}\right)^n\right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{2n^2+1}{n^2+1} = \lim_{n\to\infty} \frac{2+\frac{1}{n^2}}{1+\frac{1}{n^2}} = 2,$$

hence the series diverges!

Example 6.4.14. Consider $\sum_{n=1}^{\infty} \frac{1}{n}$. We have

$$\lim_{n\to\infty} \left|\frac{1}{n}\right|^{\frac{1}{n}} = 1,$$

so we conclude *nothing!*

Example 6.4.15. Consider $\sum_{n=1}^{\infty} \frac{n^n}{3\cdot(27^n)}$. To apply the root test, we compute

$$\lim_{n\to\infty} \left|\frac{n^n}{3\cdot(27^n)}\right|^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{1}{3}\right)^{\frac{1}{n}} \cdot \frac{n}{27} = +\infty.$$

Again, the limit diverges, as in Example 6.4.8.