NOTES for 2006-02-22
Midterm 2: Wednesday, March 1, 2006, at 7pm in Pepper Canyon 109
Today: 7.8: Comparison of Improper integrals
11.1: Sequences
Next 11.2 Series

Example 5.7.7. Compute $\int_{-1}^{3} \frac{1}{x-2} dx$. A few weeks ago you might have done this:

$$\int_{-1}^{3} \frac{1}{x-2} dx = [\ln|x-2|]_{-1}^{3} = \ln(3) - \ln(1) \qquad \text{(totally wrong!)}$$

This is not valid because the function we are integrating has a pole at $x = 2$ (see Figure 5.7.4). The integral is improper, and is only defined if both the following limits exists:

$$\lim_{t \to 2^-} \int_{-1}^{t} \frac{1}{x-2} dx \qquad \text{and} \qquad \lim_{t \to 2^+} \int_{t}^{3} \frac{1}{x-2} dx.$$

However, the limits diverge, e.g.,

$$\lim_{t \to 2^+} \int_{t}^{3} \frac{1}{x-2} dx = \lim_{t \to 2^+} (\ln|1| - \ln|t-2|) = -\lim_{t \to 2^+} \ln|t-2| = -\infty.$$

Thus $\int_{-1}^{3} \frac{1}{x-2} dx$ is divergent.

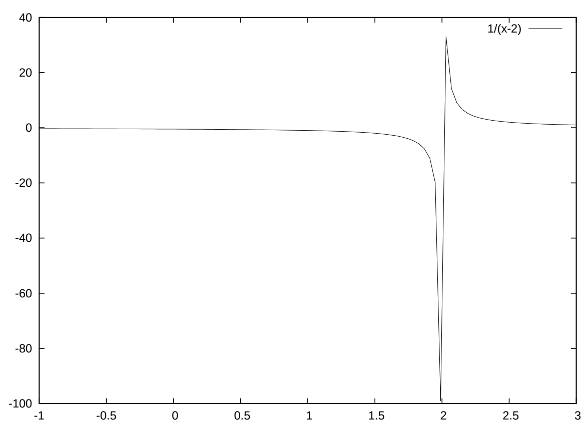

Figure 5.7.4: Graph of $\frac{1}{x-2}$

5.7.1 Convergence, Divergence, and Comparison

In this section we discuss using comparison to determine if an improper integrals converges or diverges. Recall that if f and g are continuous functions on an interval $[a, b]$ and $g(x) \leq f(x)$, then

$$\int_{a}^{b} g(x) dx \leq \int_{a}^{b} f(x) dx.$$

This observation can be *incredibly useful* in determining whether or not an improper integral converges.

Not only does this technique help in determing whether integrals converge, but it also gives you some information about their values, which is often much easier to obtain than computing the exact integral.

Theorem 5.7.8 (Comparison Theorem (special case)). *Let f and g be continuous functions with $0 \leq g(x) \leq f(x)$ for $x \geq a$.*

1. *If $\int_a^\infty f(x)dx$ converges, then $\int_a^\infty g(x)dx$ converges.*
2. *If $\int_a^\infty g(x)dx$ diverges then $\int_a^\infty f(x)dx$ diverges.*

Proof. Since $g(x) \geq 0$ for all x, the function

$$G(t) = \int_a^t g(x)dx$$

is a non-decreasing function. If $\int_a^\infty f(x)dx$ converges to some value B, then for any $t \geq a$ we have

$$G(t) = \int_a^t g(x)dx \leq \int_a^t f(x)dx \leq B.$$

Thus in this case $G(t)$ is a non-decreasing function bounded above, hence the limit $\lim_{t\to\infty} G(t)$ exists. This proves the first statement.

Likewise, the function

$$F(t) = \int_a^t f(x)dx$$

is also a non-decreasing function. If $\int_a^\infty g(x)dx$ diverges then the function $G(t)$ defined above is still non-decreasing and $\lim_{t\to\infty} G(t)$ does not exist, so $G(t)$ is not bounded. Since $g(x) \leq f(x)$ we have $G(t) \leq F(t)$ for all $\geq a$, hence $F(t)$ is also unbounded, which proves the second statement. □

The theorem is very intuitive if you think about areas under a graph. "If the bigger integral converges then so does the smaller one, and if the smaller one diverges so does the bigger ones."

Example 5.7.9. Does $\int_0^\infty \frac{\cos^2(x)}{1+x^2}dx$ converge? Answer: YES.

Since $0 \leq \cos^2(x) \leq 1$, we really do have

$$0 \leq \frac{\cos^2(x)}{1+x^2} \leq \frac{1}{1+x^2},$$

as illustrated in Figure 5.7.5. Thus

$$\int_0^\infty \frac{1}{1+x^2}dx = \lim_{t\to\infty} \tan^{-1}(t) = \frac{\pi}{2},$$

so $\int_0^\infty \frac{\cos^2(x)}{1+x^2}dx$ converges.

But why did we use $\frac{1}{1+x^2}$? It's a *guess* that turned out to work. You could have used something else, e.g., $\frac{c}{x^2}$ for some constant c. This is an illustration of how in mathematics sometimes you have to use your imagination or guess and see what happens. Don't get anxious—instead, relax, take a deep breath and explore.

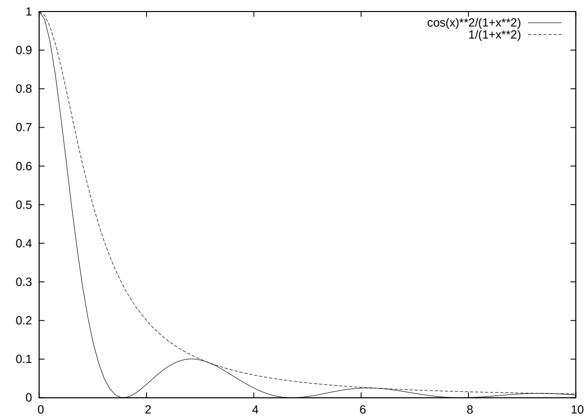

Figure 5.7.5: Graph of $\frac{\cos(x)^2}{1+x^2}$ and $\frac{1}{1+x^2}$

For example, alternatively we could have done the following:

$$\int_1^\infty \frac{\cos^2(x)}{1+x^2}dx \le \int_1^\infty \frac{1}{x^2}dx = 1,$$

and this works just as well, since $\int_0^1 \frac{\cos^2(x)}{1+x^2}dx$ converges (as $\frac{\cos^2(x)}{1+x^2}$ is continuous).

Example 5.7.10. Consider $\int_0^\infty \frac{1}{x+e^{-2x}}dx$. Does it converge or diverge? For large values of x, the term e^{-2x} very quickly goes to 0, so we expect this to diverge, since $\int_1^\infty \frac{1}{x}dx$ diverges. For $x \ge 0$, we have $e^{-2x} \le 1$, so for all x we have

$$\frac{1}{x+e^{-2x}} \ge \frac{1}{x+1} \qquad \text{(verify by cross multiplying)}.$$

But

$$\int_1^\infty \frac{1}{x+1}dx = \lim_{t\to\infty} [\ln(x+1)]_1^t = \infty$$

Thus $\int_0^\infty \frac{1}{x+e^{-2x}}dx$ must also diverge.

Note that there is a natural analogue of Theorem 5.7.8 for integrals of functions that "blow up" at a point, but we will not state it formally.

Example 5.7.11. Consider

$$\int_0^1 \frac{e^{-x}}{\sqrt{x}}dx = \lim_{t\to 0^+} \int_t^1 \frac{e^{-x}}{\sqrt{x}}dx.$$

We have

$$\frac{e^{-x}}{\sqrt{x}} \le \frac{1}{\sqrt{x}}.$$

(Coming up with this comparison might take some work, imagination, and trial and error.) We have

$$\int_0^1 \frac{e^{-x}}{\sqrt{x}}dx \le \int_0^1 \frac{1}{\sqrt{x}}dx = \lim_{t\to 0^+} 2\sqrt{1} - 2\sqrt{t} = 2.$$

thus $\int_0^1 \frac{e^{-x}}{\sqrt{x}} dx$ converges, even though we haven't figured out its value. We just know that it is ≤ 2. (In fact, it is $1.493648265\ldots$.)

What if we found a function that is bigger than $\frac{e^{-x}}{\sqrt{x}}$ and its integral diverges?? So what! This does nothing for you. Bzzzt. Try again.

Example 5.7.12. Consider the integral

$$\int_0^1 \frac{e^{-x}}{x} dx.$$

This is an improper integral since $f(x) = \frac{e^{-x}}{x}$ has a pole at $x = 0$. Does it converge? NO.
On the interal $[0, 1]$ we have $e^{-x} \geq e^{-1}$. Thus

$$\begin{aligned}
\lim_{t\to 0^+} \int_t^1 \frac{e^{-x}}{x} dx &\geq \lim_{t\to 0^+} \int_t^1 \frac{e^{-1}}{x} dx \\
&= e^{-1} \cdot \lim_{t\to 0^+} \int_t^1 \frac{1}{x} dx \\
&= e^{-1} \cdot \lim_{t\to 0^+} \ln(1) - \ln(t) = +\infty
\end{aligned}$$

Thus $\int_0^1 \frac{e^{-x}}{x} dx$ diverges.