5.7 Improper Integrals

Exam 2 Wed Mar 1: 7pm-7:50pm in ??
Today: 7.8 Improper Integrals
Monday – president's day holiday (and almost my bday)
Next — 11.1 sequences

Example 5.7.1. Make sense of $\int_0^\infty e^{-x}dx$. The integrals

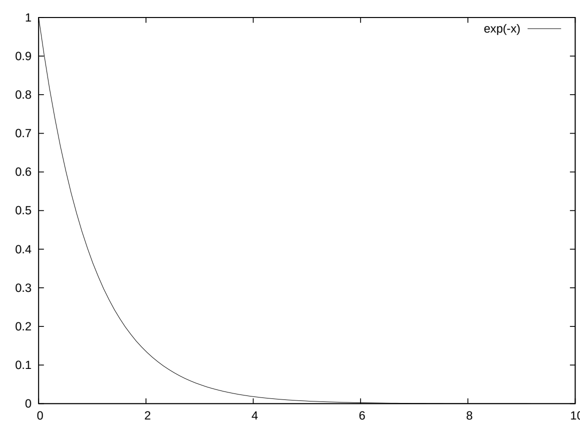
$$\int_0^t e^{-x}dx$$

make sense for each real number t. So consider

$$\lim_{t\to\infty}\int_0^t e^{-x}dx = \lim_{t\to\infty}[-e^{-x}]_0^t = 1.$$

Geometrically the area under the whole curve is the limit of the areas for finite values of t.

Figure 5.7.1: Graph of e^{-x}



Example 5.7.2. Consider $\int_0^1 \frac{1}{\sqrt{1-x^2}}dx$ (see Figure 5.7.2). Problem: The denominator of the integrand tends to 0 as x approaches the upper endpoint. Define

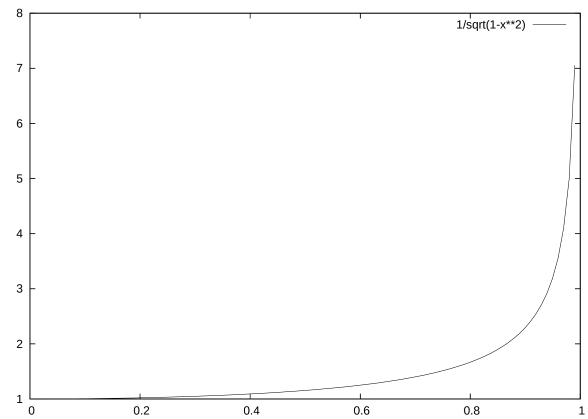
$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{1-x^2}}dx &= \lim_{t\to 1^-}\int_0^t \frac{1}{\sqrt{1-x^2}}dx \\ &= \lim_{t\to 1^-}\left(\sin^{-1}(t)-\sin^{-1}(0)\right) = \sin^{-1}(1) = \frac{\pi}{2}\end{aligned}$$

Here $t\to 1^-$ means the limit as t tends to 1 *from the left*.

Example 5.7.3. There can be multiple points at which the integral is improper. For example, consider

$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx.$$

Figure 5.7.2: Graph of $\frac{1}{\sqrt{1-x^2}}$



A crucial point is that we take the limit for the left and right endpoints independently. We use the point 0 (for convenience only!) to break the integral in half.

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx &= \int_{-\infty}^{0} \frac{1}{1+x^2}dx + \int_{0}^{\infty} \frac{1}{1+x^2}dx \\ &= \lim_{s\to-\infty}\int_{s}^{0} \frac{1}{1+x^2}dx + \lim_{t\to\infty}\int_{0}^{t} \frac{1}{1+x^2}dx \\ &= \lim_{s\to-\infty}(\tan^{-1}(0) - \tan^{-1}(s)) + \lim_{t\to\infty}(\tan^{-1}(t) - \tan^{-1}(0)) \\ &= \lim_{s\to-\infty}(-\tan^{-1}(s)) + \lim_{t\to\infty}(\tan^{-1}(t)) \\ &= -\frac{-\pi}{2} + \frac{\pi}{2} = \pi.\end{aligned}$$

The graph of $\tan^{-1}(x)$ is in Figure 5.7.3.

Example 5.7.4. Brian Conrad's paper on impossibility theorems for elementary integration begins: "The Central Limit Theorem in probability theory assigns a special significance to the cumulative area function

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-u^2}udu$$

under the Gaussian bell curve

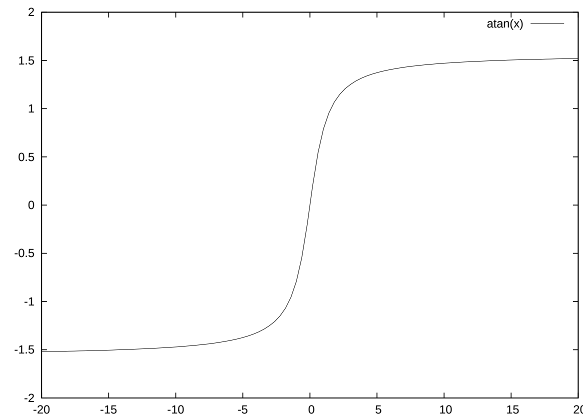
$$y = \frac{1}{\sqrt{2\pi}} \cdot e^{-u^2/2}.$$

It is known that $\Phi(\infty) = 1$."

What does this last statement *mean*? It means that

$$\lim_{t\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-t}^{0} e^{-u^2}udu + \lim_{x\to\infty}\frac{1}{\sqrt{2\pi}}\int_{0}^{x} e^{-u^2}udu = 1.$$

Figure 5.7.3: Graph of $\tan^{-1}(x)$



Example 5.7.5. Consider $\int_{-\infty}^{\infty} x dx$. Notice that

$$\int_{-\infty}^{\infty} x dx = \lim_{s \to -\infty} \int_s^0 x dx + \lim_{t \to \infty} \int_0^t x dx.$$

This diverges since each factor diverges independtly. But notice that

$$\lim_{t \to \infty} \int_{-t}^{t} x dx = 0.$$

This is *not* what $\int_{-\infty}^{\infty} x dx$ means (in this course – in a later course it could be interpreted this way)! This illustrates the importance of treating each bad point separately (since Example 5.7.3) doesn't.

Example 5.7.6. Consider $\int_{-1}^{1} \frac{1}{\sqrt[3]{x}} dx$. We have

$$\begin{aligned}\int_{-1}^{1} \frac{1}{\sqrt[3]{x}} dx &= \lim_{s \to 0^-} \int_{-1}^{s} x^{-\frac{1}{3}} dx \; + \; \lim_{t \to 0^+} \int_t^1 x^{-\frac{1}{3}} dx \\ &= \lim_{s \to 0^-} \left(\frac{3}{2} s^{\frac{2}{3}} - \frac{3}{2} \right) + \lim_{t \to 0^+} \left(\frac{3}{2} - \frac{3}{2} t^{\frac{2}{3}} \right) = 0.\end{aligned}$$

This illustrates how to be careful and break the function up into two pieces when there is a discontinuity.