EXAM 1: Wednesday 7:00-7:50pm in Pepper Canyon 109 (!)
Today: Supplement 1 (get online; also homework online)
Wednesday: Review
Bulletin board, online chat, directory, etc. – see main course website.
Review day – I will prepare no LECTURE; instead I will answer questions.
Your *job* is to have your most urgent questions ready to go!
Office hours moved: NOT Tue 11-1 (since nobody ever comes then and I'll be at a conference); instead I'll be in my office to answer questions WED 1:30-4pm, and after class on WED too.
Office: AP&M 5111

Quick review:
Given a point (x, y) in the plane, we can also view it as $x + iy$ or in polar form as $r(\cos(\theta) + i\sin(\theta))$. Polar form is great since it's good for multiplication, powering, and for extracting roots:

$$r_1(\cos(\theta_1) + i\sin(\theta_1))r_2(\cos(\theta_2) + i\sin(\theta_2)) = (r_1r_2)(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)).$$

(If you divide, you subtract the angle.) The point is that the polar form *works better* with multiplication than the rectangular form.

Theorem 4.3.6 (De Moivre's). *For any integer n we have*

$$(r(\cos(\theta) + i\sin(\theta)))^n = r^n(\cos(n\theta) + i\sin(n\theta)).$$

Since we know how to raise a complex number in polar form to the nth power, we can find all numbers with a given power, hence find the nth roots of a complex number.

Proposition 4.3.7 (nth roots). *A complex number $z = r(\cos(\theta) + i\sin(\theta))$ has n distinct nth roots:*

$$r^{1/n}\left(\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right),$$

for $k = 0, 1, \ldots, n-1$. Here $r^{1/n}$ is the real positive n-th root of r.

As a double-check, note that by De Moivre, each number listed in the proposition has nth power equal to z.

An application of De Moivre is to computing $\sin(n\theta)$ and $\cos(n\theta)$ in terms of $\sin(\theta)$ and $\cos(\theta)$. For example,

$$\begin{aligned}\cos(3\theta) + i\sin(3\theta) &= (\cos(\theta) + i\sin(\theta))^3\\ &= (\cos(\theta)^3 - 3\cos(\theta)\sin(\theta)^2) + i(3\cos(\theta)^2\sin(\theta) - \sin(\theta)^3)\end{aligned}$$

Equate real and imaginary parts to get formulas for $\cos(3\theta)$ and $\sin(3\theta)$. In the next section we will discuss going in the other direction, i.e., writing powers of sin and cos in terms of sin and cosine.

Example 4.3.8. Find the cube roots of 2.
Solution. Write 2 in polar form as

$$2 = 2(\cos(0) + i\sin(0)).$$

Then the three cube roots of 2 are

$$2^{1/3}(\cos(2\pi k/3) + i\sin(2\pi k/3)),$$

for $k = 0, 1, 2$. I.e.,

$$2^{1/3}, \quad 2^{1/3}(-1/2 + i\sqrt{3}/2), \quad 2^{1/3}(-1/2 - i\sqrt{3}/2).$$

4.4 Complex Exponentials and Trig Identities

Recall that

$$r_1(\cos(\theta_1) + i\sin(\theta_1))r_2(\cos(\theta_2) + i\sin(\theta_2)) = (r_1r_2)(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)).$$

The angles add. You've seen something similar before:

$$e^a e^b = a^{a+b}.$$

This connection between exponentiation and (4.4) gives us an idea!

If $z = x + iy$ is a complex number, *define*

$$e^z = e^x(\cos(y) + i\sin(y)).$$

We have just written polar coordinates in another form. It's a shorthand for the polar form of a complex number:

$$r(\cos(\theta) + i\sin(\theta)) = re^{i\theta}.$$

Theorem 4.4.1. *If z_1, z_2 are two complex numbers, then*

$$e^{z_1}e^{z_2} = e^{z_1+z_2}$$

Proof.

$$\begin{aligned} e^{z_1}e^{z_2} &= e^{a_1}(\cos(b_1) + i\sin(b_1)) \cdot e^{a_2}(\cos(b_2) + i\sin(b_2)) \\ &= e^{a_1+a_2}(\cos(b_1 + b_2) + i\sin(b_1 + b_2)) \\ &= e^{z_1+z_2}. \end{aligned}$$

Here we have just used (4.4). □

The following theorem is amazing, since it involves calculus.

Theorem 4.4.2. *If w is a complex number, then*

$$\frac{d}{dx}e^{wx} = we^{wx},$$

for x real. In fact, this is even true for x a complex variable (but we haven't defined differentiation for complex variables yet).

Proof. Write $w = a + bi$.

$$\begin{aligned} \frac{d}{dx}e^{wx} &= \frac{d}{dx}e^{ax+bix} \\ &= \frac{d}{dx}(e^{ax}(\cos(bx) + i\sin(bx))) \\ &= \frac{d}{dx}(e^{ax}\cos(bx) + ie^{ax}\sin(bx)) \\ &= \frac{d}{dx}(e^{ax}\cos(bx)) + i\frac{d}{dx}(e^{ax}\sin(bx)) \end{aligned}$$

Now we use the product rule to get

$$\begin{aligned}\frac{d}{dx}(e^{ax}\cos(bx)) + i\frac{d}{dx}(e^{ax}\sin(bx))&\\ &= ae^{ax}\cos(bx) - be^{ax}\sin(bx) + i(ae^{ax}\sin(bx) + be^{ax}\cos(bx))\\ &= e^{ax}(a\cos(bx) - b\sin(bx) + i(a\sin(bx) + b\cos(bx))\end{aligned}$$

On the other hand,

$$\begin{aligned}we^{wx} &= (a+bi)e^{ax+bxi}\\ &= (a+bi)e^{ax}(\cos(bx) + i\sin(bx))\\ &= e^{ax}(a+bi)(\cos(bx) + i\sin(bx))\\ &= e^{ax}((a\cos(bx) - b\sin(bx)) + i(a\sin(bx)) + b\cos(bx))\end{aligned}$$

Wow!! We did it! □

That Theorem 4.4.2 is true is pretty amazing. It's what really gets complex analysis going.

Example 4.4.3. Here's another fun fact: $e^{i\pi} + 1 = 0$.
Solution. By definition, have $e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1 + i0 = -1$.

4.4.1 Trigonometry and Complex Exponentials

Amazingly, trig functions can also be expressed back in terms of the complex exponential. Then *everything* involving trig functions can be transformed into something involving the exponential function. This is very surprising.

In order to easily obtain trig identities like $\cos(x)^2 + \sin(x)^2 = 1$, let's write $\cos(x)$ and $\sin(x)$ as complex exponentials. From the definitions we have

$$e^{ix} = \cos(x) + i\sin(x),$$

so

$$e^{-ix} = \cos(-x) + i\sin(-x) = \cos(x) - i\sin(x).$$

Adding these two equations and dividing by 2 yields a formula for $\cos(x)$, and subtracting and dividing by $2i$ gives a formula for $\sin(x)$:

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2} \qquad \sin(x) = \frac{e^{ix} - e^{-ix}}{2i}.$$

We can now derive trig identities. For example,

$$\begin{aligned}\sin(2x) &= \frac{e^{i2x} - e^{-i2x}}{2i}\\ &= \frac{(e^{ix} - e^{-ix})(e^{ix} + e^{-ix})}{2i}\\ &= 2\frac{e^{ix} - e^{-ix}}{2i}\frac{e^{ix} + e^{-ix}}{2} = 2\sin(x)\cos(x).\end{aligned}$$

I'm unimpressed, given that you can get this much more directly using

$$(\cos(2x) + i\sin(2x)) = (\cos(x) + i\sin(x))^2 = \cos^2(x) - \sin^2(x) + i2\cos(x)\sin(x),$$

and equating imaginary parts. But there are more interesting examples.

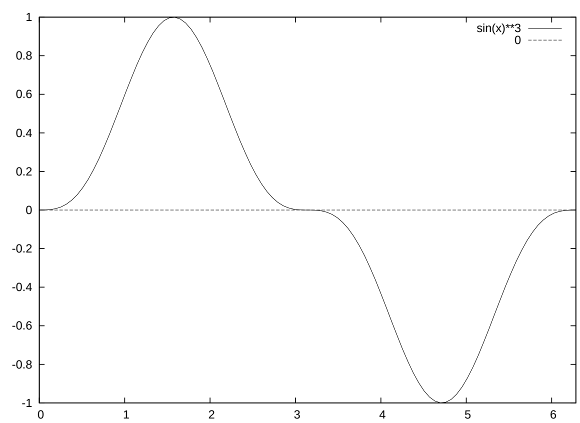
Next we verify that (4.4.1) implies that $\cos(x)^2 + \sin(x)^2 = 1$. We have

$$\begin{aligned} 4(\cos(x)^2 + \sin(x)^2) &= \left(e^{ix} + e^{-ix}\right)^2 + \left(\frac{e^{ix} - e^{-ix}}{i}\right)^2 \\ &= e^{2ix} + 2 + e^{-2ix} - (e^{2ix} - 2 + e^{-2ix}) = 4. \end{aligned}$$

The equality just appears as a follow-your-nose algebraic calculation.

Example 4.4.4. Compute $\sin(x)^3$ as a sum of sines and cosines with no powers.
Solution. We use (4.4.1):

Figure 4.4.1: What is $\sin(x)^3$?



$$\begin{aligned} \sin(x)^3 &= \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^3 \\ &= \left(\frac{1}{2i}\right)^3 (e^{ix} - e^{-ix})^3 \\ &= \left(\frac{1}{2i}\right)^3 (e^{ix} - e^{-ix})(e^{ix} - e^{-ix})(e^{ix} - e^{-ix}) \\ &= \left(\frac{1}{2i}\right)^3 (e^{ix} - e^{-ix})(e^{2ix} - 2 + e^{-2ix}) \\ &= \left(\frac{1}{2i}\right)^3 (e^{3ix} - 2e^{ix} + e^{-ix} - e^{ix} + 2e^{-ix} - e^{-3ix}) \\ &= \left(\frac{1}{2i}\right)^3 ((e^{3ix} - e^{-3ix}) - 3(e^{ix} - e^{-ix})) \\ &= -\left(\frac{1}{4}\right)\left[\frac{e^{3ix} - e^{-3ix}}{2i} - 3\cdot\frac{e^{ix} - e^{-ix}}{2i}\right] \\ &= \frac{3\sin(x) - \sin(3x)}{4}. \end{aligned}$$