4.3 Complex Numbers

A complex number is an expression of the form $a+bi$, where a and b are real numbers, and $i^2=-1$. We add and multiply complex numbers as follows:

$$(a+bi)+(c+di)=(a+c)+(b+d)i$$
$$(a+bi)\cdot(c+di)=(ac-bd)+(ad+bc)i$$

The complex conjugate of a complex number is

$$\overline{a+bi}=a-bi.$$

Note that

$$(a+bi)(\overline{a+bi})=a^2+b^2$$

is a real number (has no complex part).

If $c+di\neq 0$, then

$$\frac{a+bi}{c+di}=\frac{(a+bi)(c-di)}{c^2+d^2}=\frac{1}{c^2+d^2}((ac+bd)+(bc-ad)i).$$

Example 4.3.1. $(1-2i)(8-3i)=2-19i$ and $1/(1+i)=(1-i)/2=1/2-(1/2)i$.

Complex numbers are incredibly useful in providing better ways to understand ideas in calculus, and more generally in many applications (e.g., electrical engineering, quantum mechanics, fractals, etc.). For example,

- Every polynomial $f(x)$ **factors** as a product of linear factors $(x-\alpha)$, if we allow the α's in the factorization to be complex numbers. For example,

  $$f(x)=x^2+1=(x-i)(x+i).$$

  This will provide an easier to use variant of the "partial fractions" integration technique, which we will see later.

- Complex numbers are in **correspondence** with points in the plane via $(x,y)\leftrightarrow x+iy$. Via this correspondence we obtain a way to add and *multiply* points in the plane.

- Similarly, points in **polar coordinates** correspond to complex numbers:

  $$(r,\theta)\leftrightarrow r(\cos(\theta)+i\sin(\theta)).$$

- Complex numbers provide a very nice way to remember and **understand trig identities**.

4.3.1 Polar Form

The *polar form* of a complex number $x+iy$ is $r(\cos(\theta)+i\sin(\theta))$ where (r,θ) are any choice of polar coordinates that represent the point (x,y) in rectangular coordinates. Recall that you can find the polar form of a point using that

$$r=\sqrt{x^2+y^2}\quad\text{and}\quad\theta=\tan^{-1}(y/x).$$

NOTE: The "existence" of complex numbers wasn't generally accepted until people got used to a geometric interpretation of them.

Example 4.3.2. Find the polar form of $1+i$.
Solution. We have $r=\sqrt{2}$, so

$$1+i=\sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)=\sqrt{2}\left(\cos(\pi/4)+i\sin(\pi/4)\right).$$

Example 4.3.3. Find the polar form of $\sqrt{3}-i$.
Solution. We have $r=\sqrt{3+1}=2$, so

$$\sqrt{3}-i=2\left(\frac{\sqrt{3}}{2}+i\frac{-1}{2}\right)=2\left(\cos(-\pi/6)+i\sin(-\pi/6)\right)$$

[[A picture is useful here.]]

Finding the polar form of a complex number is exactly the same problem as finding polar coordinates of a point in rectangular coordinates. The only hard part is figuring out what θ is.

If we write complex numbers in rectangular form, their sum is easy to compute:

$$(a+bi)+(c+di)=(a+c)+(b+d)i$$

The beauty of polar coordinates is that if we write two complex numbers in polar form, then their *product* is very easy to compute:

$$r_1(\cos(\theta_1)+i\sin(\theta_1))\cdot r_2(\cos(\theta_2)+i\sin(\theta_2))=(r_1r_2)(\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)).$$

The magnitudes multiply and the angles add. The above formula is true because of the double angle identities for sin and cos (and it is how I remember those formulas!).

$$\begin{aligned}&(\cos(\theta_1)+i\sin(\theta_1))\cdot(\cos(\theta_2)+i\sin(\theta_2))\\&\qquad=(\cos(\theta_1)\cos(\theta_2)-\sin(\theta_1)\sin(\theta_2))+i(\sin(\theta_1)\cos(\theta_2)+\cos(\theta_1)\sin(\theta_2)).\end{aligned}$$

For example, the power of a singular complex number in polar form is easy to compute; just power the r and multiply the angle.

Theorem 4.3.4 (De Moivre's). *For any integer n we have*

$$(r(\cos(\theta)+i\sin(\theta)))^n=r^n(\cos(n\theta)+i\sin(n\theta)).$$

Example 4.3.5. Compute $(1+i)^{2006}$.
Solution. We have

$$\begin{aligned}(1+i)^{2006}&=(\sqrt{2}\left(\cos(\pi/4)+i\sin(\pi/4)\right))^{2006}\\&=\sqrt{2}^{2006}\left(\cos(2006\pi/4)+i\sin(2006\pi/4)\right)\\&=2^{1003}\left(\cos(3\pi/2)+i\sin(3\pi/2)\right)\\&=-2^{1003}i\end{aligned}$$

To get $\cos(2006\pi/4)=\cos(3\pi/2)$ we use that $2006/4=501.5$, so by periodicity of cosine, we have

$$\cos(2006\pi/4)=\cos((501.5)\pi-250(2\pi))=\cos(1.5\pi)=\cos(3\pi/2).$$

Another application of De Moivre is to computing $\sin(n\theta)$ and $\cos(n\theta)$ in terms of $\sin(\theta)$ and $\cos(\theta)$. For example,

$$\begin{aligned}\cos(3\theta) + i\sin(3\theta) &= (\cos(\theta) + i\sin(\theta))^3 \\ &= (\cos(\theta)^3 - 3\cos(\theta)\sin(\theta)^2) + i(3\cos(\theta)^2\sin(\theta) - \sin(\theta)^3)\end{aligned}$$

Equate real and imaginary parts to get formulas for $\cos(3\theta)$ and $\sin(3\theta)$.

Since we know how to raise a complex number in polar form to the n power, we can find all numbers with a given power, hence find the nth roots of a complex number.

Proposition 4.3.6 (nth roots). *A complex number $z = r(\cos(\theta) + i\sin(\theta))$ has n distinct nth roots:*

$$r^{1/n}\left(\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right),$$

for $k = 0, 1, \ldots, n-1$. Here $r^{1/n}$ is the real positive n-th root of r.

As a double-check, note that by De Moivre, each number listed in the proposition has nth power equal to z.

Example 4.3.7. Find the cube roots of 2.
Solution. Write 2 in polar form as

$$2 = 2(\cos(0) + i\sin(0)).$$

Then the three cube roots of 2 are

$$2^{1/3}(\cos(2\pi k/3) + i\sin(2\pi k/3)),$$

for $k = 0, 1, 2$. I.e.,

$$2^{1/3}, \quad 2^{1/3}(-1/2 + i\sqrt{3}/2), \quad 2^{1/3}(-1/2 - i\sqrt{3}/2).$$

4.4 Complex Exponentials and Trig Identities

If $z = a + ib$ is a complex number, define

$$e^z = e^a(\cos(b) + i\sin(b)).$$

This has all the right properties. E.g.,

$$e^{z_1}e^{z_2} = e^{z_1+z_2}$$

since

$$\begin{aligned}e^{z_1}e^{z_2} &= e_1^a(\cos(b_1) + i\sin(b_1)) \cdot e_2^a(\cos(b_2) + i\sin(b_2)) \\ &= e^{a_1+a_2}(\cos(b_1 + b_2) + i\sin(b_1 + b_2)) \\ &= e^{z_1+z_2}.\end{aligned}$$

Here we have just used our observation from the previous section about how to multiply complex numbers in polar coordinates.

In order to easily obtain trig identities like $\cos(x)^2 + \sin(x)^2 = 1$, let's write $\cos(x)$ and $\sin(x)$ as a complex exponential. From the definitions we have

$$e^{ix} = \cos(x) + i\sin(x),$$

and

$$e^{-ix} = \cos(-x) + i\sin(-x) = \cos(x) - i\sin(x).$$

Adding these two equations and dividing by 2 yields a formula for $\cos(x)$, and subtracting and dividing by $2i$ gives a formula for $\sin(x)$:

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2} \qquad \sin(x) = \frac{e^{ix} - e^{-ix}}{2i}.$$

We can now derive trig identities. For example,

$$\begin{aligned}\sin(2x) &= \frac{e^{i2x} - e^{-i2x}}{2i} \\ &= \frac{(e^{ix} - e^{-ix})(e^{ix} + e^{-ix})}{2i} \\ &= 2\frac{e^{ix} - e^{-ix}}{2i}\frac{e^{ix} + e^{-ix}}{2} \\ &= 2\sin(x)\cos(x).\end{aligned}$$

Remark 4.4.1. Frankly, I'm unimpressed, given that you can get this much more directly using

$$\begin{aligned}(\cos(2x) + i\sin(2x)) &= (\cos(x) + i\sin(x))^2 \\ &= \cos^2(x) - \sin^2(x) + i2\cos(x)\sin(x)\end{aligned}$$

and equating imaginary parts.

Example 4.4.2. We have $e^{i\pi} + 1 = 0$.
Solution. By definition, have $e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1 + i0 = -1$.