2.3 Average Values

Quiz Answers: (1) 29, (2) $\frac{1}{2}\ln\left|x^2+1\right| + \tan^{-1}(x)$
Exam 1: Wednesday, Feb 1, 7:00pm–7:50pm, here.
Today: §6.5 – Average Values
Today: §10.3 – Polar coords
NEXT: §10.4 –Areas in Polar coords

Why did we skip from §6.5 to §10.3? Later we'll go back and look at trig functions and complex exponentials; these ideas will fit together more than you might expect. We'll go back to §7.1 on Feb 3.

In this section we use Riemann sums to extend the familiar notion of an average, which provides yet another physical interpretation of integration.

Recall: Suppose $y_1, \ldots, y_n$ are the amount of rain each day in La Jolla, since you moved here. The average rainful per day is

$$y_{\text{avg}} = \frac{y_1 + \cdots + y_n}{n} = \frac{1}{n}\sum_{i=1}^{n} y_i.$$

Definition 2.3.1 (Average Value of Function). Suppose f is a continuous function on an interval $[a, b]$. The *average value* of f on $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)dx.$$

Motivation: If we sample f at n points x_i, then

$$f_{\text{avg}} \sim \frac{1}{n}\sum_{i=1}^{n} f(x_i) = \frac{(b-a)}{n(b-a)}\sum_{i=1}^{n} f(x_i) = \frac{1}{(b-a)}\sum_{i=1}^{n} f(x_i)\Delta x,$$

since $\Delta x = \dfrac{b-a}{n}$. This is a Riemann sum!

$$\frac{1}{(b-a)}\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \frac{1}{(b-a)}\int_a^b f(x)dx.$$

This explains why we defined f_{avg} as above.

Example 2.3.2. What is the average value of $\sin(x)$ on the interval $[0, \pi]$?

$$\begin{aligned}\frac{1}{\pi-0}\int_0^{\pi}\sin(x)dx &= \frac{1}{\pi-0}\Big[-\cos(x)\Big]_0^{\pi} \\ &= \frac{1}{\pi}\Big[-(-1)-(-1)\Big]_0^{\pi} = \frac{2}{\pi}\end{aligned}$$

Observation: If you multiply both sides by $(b-a)$ in Definition 2.3.1, you see that the average value times the length of the interval is the area, i.e., the average value gives you a rectangle with the same area as the area under your function. In particular, in Figure 2.3.1 the area between the x-axis and $\sin(x)$ is exactly the same as the area between the horizontal line of height $2/\pi$ and the x-axis.

Figure 2.3.1: What is the average value of $\sin(x)$?

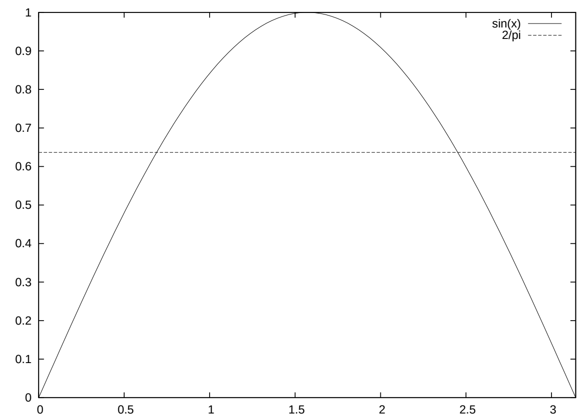
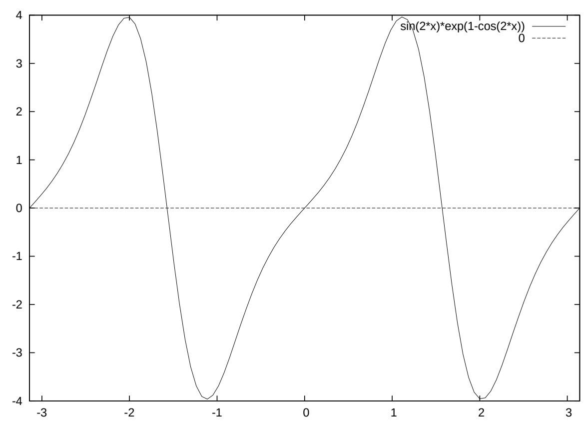


Figure 2.3.2: What is the average value?



Example 2.3.3. What is the average value of $\sin(2x)e^{1-\cos(2x)}$ on the interval $[-\pi, \pi]$?

$$\frac{1}{\pi - (-\pi)} \int_{-\pi}^{\pi} \sin(2x)e^{1-\cos(2x)} dx = 0 \qquad \text{(since the function is odd!)}$$

Theorem 2.3.4 (Mean Value Theorem). *Suppose f is a continuous function on $[a, b]$. Then there is a number c in $[a, b]$ such that $f(c) = f_{\text{avg}}$.*

This says that f assumes its average value. It is a used very often in understanding why certain statements are true. Notice that in Examples 2.3.2 and 2.3.3 it is just the assertion that the graphs of the function and the horizontal line interesect.

Proof. Let $F(x) = \int_a^x f(t)dt$. Then $F'(x) = f(x)$. By the mean value theorem for derivatives, there is $c \in [a, b]$ such that $f(c) = F'(c) = (F(b) - F(a))/(b - a)$. But by the fundamental theorem of calculus,

$$f(c) = \frac{F(b) - F(a)}{b - a} = \frac{1}{b - a} \int_a^b f(x)dx = f_{\text{avg}}.$$

$\square$

2.4 Polar Coordinates

Rectangular coordinates allow us to describe a point (x, y) in the plane in a different way, namely

$$(x, y) \leftrightarrow (r, \theta),$$

where r is any real number and θ is an angle.

Polar coordinates are extremely useful, especially when thinking about complex numbers. Note, however, that the (r, θ) representation of a point is very non-unique.

First, θ is not determined by the point. You could add 2π to it and get the same point:

$$\left(2, \frac{\pi}{4}\right) = \left(2, \frac{9\pi}{4}\right) = \left(2, \frac{\pi}{4} + 389 \cdot 2\pi\right). = \left(2, \frac{-7\pi}{4}\right)$$

Also that r can be negative introduces further non-uniqueness:

$$\left(1, \frac{\pi}{2}\right) = \left(-1, \frac{3\pi}{2}\right).$$

Think about this as follows: facing in the direction $3\pi/2$ and backing up 1 meter gets you to the same point as looking in the direction $\pi/2$ and walking forward 1 meter.

We can convert back and forth between cartesian and polar coordinates using that

$$x = r\cos(\theta) \tag{2.4.1}$$
$$y = r\sin(\theta), \tag{2.4.2}$$

and in the other direction

$$r^2 = x^2 + y^2 \tag{2.4.3}$$
$$\tan(\theta) = \frac{y}{x} \tag{2.4.4}$$

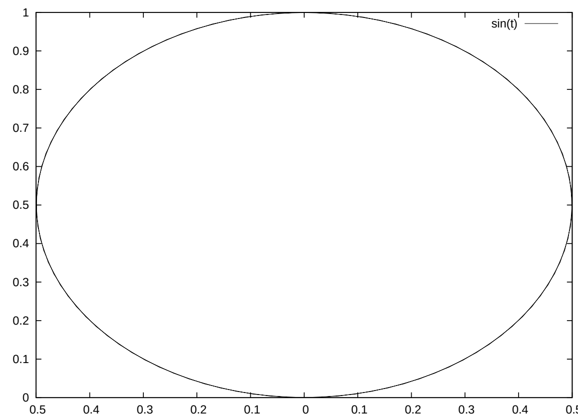
(Thus $r = \pm\sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$.)

Example 2.4.1. Sketch $r = \sin(\theta)$, which is a circle sitting on top the x axis.

We plug in points for one period of the function we are graphing—in this case $[0, 2\pi]$:

0	$\sin(0) = 0$
$\pi/6$	$\sin(\pi/6) = 1/2$
$\pi/4$	$\sin(\pi/4) = \frac{\sqrt{2}}{2}$
$\pi/2$	$\sin(\pi/2) = 1$
$3\pi/4$	$\sin(3\pi/4) = \frac{\sqrt{2}}{2}$
π	$\sin(\pi) = 0$
$\pi + \pi/6$	$\sin(\pi + \pi/6) = -1/2$

Figure 2.4.1: Graph of $r = \sin(\theta)$.



Notice it is nice to allow r to be negative, so we don't have to restrict the input. BUT it is really painful to draw this graph by hand.

To more accurately draw the graph, let's try converting the equation to one involving polar coordinates. This is easier if we multiply both sides by r:

$$r^2 = r\sin(\theta).$$

Note that the new equation has the extra solution ($r = 0, \theta =$ anything), so we have to be careful not to include this point. Now convert to cartesian coordinates using (2.4.1) to obtain (2.4.3):

$$x^2 + y^2 = y.$$

The graph of (2.4.1) is the same as that of $r = \sin(\theta)$. To confirm this we complete the square:

$$\begin{aligned} x^2 + y^2 &= y \\ x^2 + y^2 - y &= 0 \\ x^2 + (y - 1/2)^2 &= 1/4 \end{aligned}$$

Thus the graph of (2.4.1) is a circle of radius 1/2 centered at $(0, 1/2)$.

Actually *any* polar graph of the form $r = a\sin(\theta) + b\cos(\theta)$ is a circle, as you will see in homework problem 67 by generalizing what we just did.