1.2 Indefinite Integrals and Change

(William Stein, Math 20b, Winter 2006)

Homework: Do the following by Tuesday, January 17.
* Section 5.3: 13, 37, 55, 67
* Section 5.4: 2, 9, 13, 27, 33, 39, 45, 47, 51, 53
* Section 5.5: 11, 23, 31, 37, 41, 55, 57, 63, 65, 75, 79
The first quiz will be on Friday, Jan 20 and will consist of two problems from this homework.
Ace the first quiz!

1.2.1 Indefinite Integrals

The notation $\int f(x)dx = F(x)$ means that $F'(x) = f(x)$ on some (usually specified) domain of definition of $f(x)$.

Definition 1.2.1 (Anti-derivative). We call $F(x)$ an *anti-derivative* of $f(x)$.

Proposition 1.2.2. *Suppose f is a continuous function on an interval (a, b). Then any two antiderivatives differ by a constant.*

Proof. If $F_1(x)$ and $F_2(x)$ are both antiderivatives of a function $f(x)$, then

$$(F_1(x) - F_2(x))' = F_1'(x) - F_2'(x) = f(x) - f(x) = 0.$$

Thus $F_1(x) - F_2(x) = c$ from some constant c (since only constant functions have slope 0 everywhere). Thus $F_1(x) = F_2(x) + c$ as claimed. □

We thus often write

$$\int f(x)dx = F(x) + c,$$

where c is an (unspecified fixed) constant.

Note that the proposition need not be true if f is not defined on a whole interval. For example, $f(x) = 1/x$ is not defined at 0. For any pair of constants c_1, c_2, the function

$$F(x) = \begin{cases} \ln(|x|) + c_1 & x < 0, \\ \ln(x) + c_2 & x > 0, \end{cases}$$

satisfies $F'(x) = f(x)$ for all $x \neq 0$. We often still just write $\int 1/x = \ln(|x|) + c$ anyways, meaning that this formula is supposed to hold only on one of the intervals on which $1/x$ is defined (e.g., on $(-\infty, 0)$ or $(0, \infty)$).

We pause to emphasize the notation difference between definite and indefinite integration.

$$\begin{aligned} \int_a^b f(x)dx &= \text{a specific number} \\ \int f(x)dx &= \text{a (family of) functions} \end{aligned}$$

One of the *main goals* of this course is to help you to get really good at computing $\int f(x)dx$ for various functions $f(x)$. It is useful to memorize a table of examples (see, e.g., page 406 of Stewart), since often the trick to integration is to relate a given integral to a known one. Integration is like solving a puzzle or playing a game, and often you win by moving into a position where you know how to defeat your opponent, e.g., relating your integral to integrals that you already know how to do. If you know how to do a basic collection of integrals, it will be easier for you to see how to get to a known integral from an unknown one.

Whenever you successfully compute $F(x) = \int f(x)dx$, then you've constructed a *mathematical gadget* that allows you to very quickly compute $\int_a^b f(x)dx$ for any a, b (in the interval of definition of $f(x)$). The gadget is $F(b) - F(a)$. This is really powerful.

1.2.2 Examples

Example 1.2.3.

$$\begin{aligned}\int x^2 + 1 + \frac{1}{x^2+1}dx &= \int x^2dx + \int 1dx + \int \frac{1}{x^2+1}dx \\ &= \frac{1}{3}x^2 + x + \tan^{-1}(x) + c.\end{aligned}$$

Example 1.2.4.

$$\int \sqrt{\frac{5}{x}}dx = \int \sqrt{5}x^{-1/2}dx = 2\sqrt{5}x^{1/2} + c.$$

Example 1.2.5.

$$\int \frac{\sin(2x)}{\sin(x)}dx = \int \frac{2\sin(x)\cos(x)}{\sin(x)} = \int 2\cos(x) = 2\sin(x) + c$$

1.2.3 Physical Intuition

In the previous lecture we mentioned a relation between velocity, distance, and the meaning of integration, which gave you a physical way of thinking about integration. In this section we generalize our previous observation.

The following is a restatement of the fundamental theorem of calculus:

Theorem 1.2.6 (Net Change Theorem). *The definite integral of the rate of change $F'(x)$ of some quantity $F(x)$ is the net change in that quantity:*

$$\int_a^b F'(x)dx = F(b) - F(a).$$

For example, if $p(t)$ is the population of students at UCSD at time t, then $p'(t)$ is the rate of change. Lately $p'(t)$ has been positive since $p(t)$ is growing (rapidly!). The net change interpretation of integration is that

$$\int_{t_1}^{t_2} p'(t)dt = p(t_2) - p(t_1) = \text{ change in number of students from time } t_1 \text{ to } t_2.$$

Another very common example you'll seen in problems involves water flow into or out of something. If the volume of water in your bathtub is $V(t)$ gallons at time t (in seconds), then the rate at which your tub is draining is $V'(t)$ gallons per second. If you have the geekiest drain imaginable, it prints out the drainage rate $V'(t)$. You can use that printout to determine how much water drained out from time t_1 to t_2:

$$\int_{t_1}^{t_2} V'(t)dt \ = \ \text{water that drained out from time } t_1 \text{ to } t_2$$

Some problems will try to confuse you with different notions of change. A standard example is that if a car has *velocity* $v(t)$, and you drive forward, then slam it in reverse and drive backward to where you start (say 10 seconds total elapse), then $v(t)$ is positive some of the time and negative some of the time. The integral $\int_0^{10} v(t)dt$ is not the total distance registered on your odometer, since $v(t)$ is partly positive and partly negative. If you want to express how far you actually drove going back and forth, compute $\int_0^{10} |v(t)|dt$. The following example emphasizes this distinction:

Example 1.2.7. *An ancient dragon is pacing on the cliffs in Del Mar, and has velocity $v(t) = t^2 - 2t - 8$. Find (1) the* displacement *of the dragon from time $t = 1$ until time $t = 6$ (i.e., how far the dragon is at time 6 from where it was at time 1), and (2) the* total distance *the dragon paced from time $t = 1$ to $t = 6$.*

For (1), we compute

$$\int_1^6 (t^2 - 2t - 8)dt = \left[\frac{1}{3}t^3 - t^2 - 8t\right]_1^6 = -\frac{10}{3}.$$

For (2), we compute the integral of $|v(t)|$:

$$\int_1^6 |t^2 - 2t - 8|dt = \left[-\left(\frac{1}{3}t^3 - t^2 - 8t\right)\right]_1^4 + \left[\frac{1}{3}t^3 - t^2 - 8t\right]_4^6 = 18 + \frac{44}{3} = \frac{98}{3}.$$

1.2.4 Remarks on How to Learn Calculus

In order to learn Calculus it's *crucial* for you to do all the assigned problems and then some. When I was a student and started doing well in math (instead of poorly!), the key difference was that I started doing an insane number of problems (e.g., every single problem in the book). Push yourself to the limit!

Computers

I think the best way to use a computer in learning Calculus is as a sort of solutions manual, but better. Do a problem first by hand. Then *verify* correctness of your solution. This is way better than what you get by using a solutions manual!

- You can try similar problems (not in the homework) and also verify your answers. This is like playing solitaire, but is much more creative.

- You can verify key steps of what you did by hand using the computer. E.g., if you're confused about one of part of *your* approach to computing an integral, you can compare what you get with the computer. Solution manuals either give you only the solution or a particular sequence of steps to get there, which might have little to do with the brilliantly original strategy you invented.

For this course its most useful to have a program that does symbolic integration. I recommend maxima, which is a fairly simple **completely free and open source** program written (initially) in the 1960s at MIT. Download it for free from `http://maxima.sourceforge.net`

It's not insanely powerful, but it'll instantly do (something with) pretty much any integral in this class, and a lot more. Plus if you know lisp you can read the source code. (You could also buy Maple or Mathematica, or use a TI-89 calculator.)

Here are some maxima examples:

```
(%i2) integrate(x^2 + 1 + 1/(x^2+1), x);
                                                  3
                                                 x
(%o2)                             atan(x) + -- + x
                                                 3

(%i3) integrate(sqrt(5/x), x);
(%o3)                             2 sqrt(5) sqrt(x)

(%i4) integrate(sin(2*x)/sin(x), x);
(%o4)                                 2 sin(x)

(%i5) integrate(sin(2*x)/sin(x), x, 0, %pi);
(%o5)                                     0

(%i6) integrate(sin(2*x)/sin(x), x, 0, %pi/2);
(%o6)                                     2
```