CHAPTER IV

Abelian Varieties over $\mathbb{C}$

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§0. Introduction

These lecture notes present, in outline, the theory of abelian varieties over the complex numbers. They focus mainly on the analytic side of the subject. In the first section we prove some basic results on complex tori. The second section is devoted to a discussion of isogenies. The third section (the longest) describes the necessary and sufficient conditions that a complex torus must satisfy in order to be isomorphic to an abelian variety. In the fourth section we describe the construction of the dual abelian variety and the concluding two sections discuss polarizations and the moduli space of principally polarized abelian varieties. Proofs for the most part are omitted or only sketched. Details can be found in [SW] or [L–A] (see the list of references at the end of this chapter). For the algebraic–geometric study of abelian varieties over arbitrary fields, the reader is referred to [M–AV] and to the articles of J. S. Milne in this volume.

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§1. Complex Tori

An *abelian variety* A is a complete and connected algebraic group defined over the field of complex numbers. Thus A comes equipped with a multiplication $m: A \times A \to A$ and an inverse map $i: A \to A$ which are morphisms of varieties and satisfy the usual group axioms. The complex points $A(\mathbb{C})$ is then a connected, compact, complex Lie group. We will begin by considering properties of such objects. Let T be an arbitrary connected, compact, complex Lie group. Then:

(1) T is a commutative group

To see why this is so, let V denote the tangent space to T at the identity element e. Consider the adjoint representation of T on V:

$$\mathrm{Ad}: T \to \mathrm{Aut}_{\mathbb{C}}(V).$$

($\mathrm{Ad}(t)$ is the differential of the conjugation map $u \to tut^{-1}$ on T.) The coordinate functions with respect to a basis of V are holomorphic on the compact complex manifold T and so must be constants. Thus $\mathrm{Ad}(t) = \mathrm{Ad}(e) = \mathrm{Id}$ for all $t \in T$. It is now easy to check that the exponential map, $\exp: V \to T$, maps V onto a subgroup of the center of T. Since T is connected, $\exp(V)$ generates T, and so T is commutative.

(2) T is a complex torus

A more refined analysis shows exp is a surjective homomorphism from V to T with kernel Λ a discrete subgroup. Recall that a discrete subgroup of a real vector space with compact quotient is called a lattice. Thus, $T \approx V/\Lambda$, is a complex torus. See [M–AV] for the proof.

From now on we will write the group law on T additively and denote the identity element by 0.

(3) Holomorphic 1-forms

The representation of T as a complex torus can be achieved in another way. Let Ω be the vector space of holomorphic 1-forms on T. Define $H_1(T, \mathbb{Z}) \to \Omega^* = \mathrm{Hom}_{\mathbb{C}}(\Omega, \mathbb{C})$ by $\gamma \to (\omega \mapsto \int_\gamma \omega)$ where $\int_\gamma \omega$ is the integral of ω around the integral 1-cycle γ.

This map is injective and the image Λ is a lattice in Ω^*. Now define $T \to \Omega^*/\Lambda$ by $p \mapsto (\omega \to \int_0^p \omega)$. Note that $\int_0^p \omega$ is well-defined modulo Λ. This yields an isomorphism

$$T \approx \Omega^*/\Lambda.$$

(4) Mappings between complex tori

Suppose T_1 and T_2 are complex tori and $\phi: T_1 \to T_2$ is a holomorphic map. If $\phi(0) = 0$ then ϕ is a homomorphism. This is implied by (3) above since ϕ induces a linear map $\Omega_1^* \to \Omega_2^*$ which takes Λ_1 to Λ_2. In general, ϕ is a homomorphism followed by a translation. If we write $T_j = V_j/\Lambda_j$ then every holomorphic homomorphism from T_1 to T_2 is induced by a $\mathbb{C}$ linear map from V_1 to V_2 such that $\phi(\Lambda_1) \subseteq \Lambda_2$. We continue to call this map ϕ. This yields two faithful representations:

$$\rho_{\mathbb{C}}: \mathrm{Hom}(T_1, T_2) \to \mathrm{Hom}_{\mathbb{C}}(V_1, V_2),$$

$$\rho_{\mathbb{Z}}: \mathrm{Hom}(T_1, T_2) \to \mathrm{Hom}_{\mathbb{Z}}(\Lambda_1, \Lambda_2),$$

called the complex representation and the rational representation respectively. The fact that $\rho_{\mathbb{Z}}$ is faithful shows immediately that $\mathrm{Hom}(T_1, T_2)$ is a finitely generated, torsion-free, abelian group of rank $\le 4(\dim T_1)(\dim T_2)$.

The case $T_1 = T_2 = T$ is of particular interest. $\mathrm{Hom}(T, T) = \mathrm{End}(T)$ is a ring which we will discuss further below. Here we present another way of looking at endomorphisms of complex tori. Suppose Λ is a lattice in an even-dimensional real vector space V. The real torus V/Λ will be a complex torus if V has the structure of a complex vector space. A complex structure on V is given by an $\mathbb{R}$-linear map $J: V \to V$ such that $J^2 = -\mathrm{Id}$ (set $iv = Jv$). Any $\mathbb{Z}$-linear map $\phi: \Lambda \to \Lambda$ defines an $\mathbb{R}$-endomorphism of $V = \mathbb{R} \otimes_{\mathbb{Z}} \Lambda$ and therefore an endomorphism of V/Λ. The map ϕ is an endomorphism of the *complex* torus V/Λ if and only if $\phi \circ J = J \circ \phi$. Thus, $\mathrm{End}(T) = \{\phi \in \mathrm{End}_{\mathbb{Z}}(\Lambda) | \phi \circ J = J \circ \phi\}$. Continuing in this direction leads to the definition of the Hodge group, a certain $\mathbb{Q}$-algebraic subgroup of $\mathrm{Gl}(V)$ whose complex points contain J. However, we will not pursue this.

(5) The image and kernel of a morphism.

If $\phi: T_1 \to T_2$ is a morphism of complex tori then im ϕ is a subtorus of T_2 while ker ϕ is a closed subgroup of T_1 whose connected component is a subtorus of finite index in ker ϕ. Both these facts are easily established.

§2. Isogenies of Complex Tori

A morphism $\phi: T_1 \to T_2$ is an *isogeny* if it is a surjective homomorphism with finite kernel. The order of the kernel is called the *degree of* ϕ, $\deg(\phi)$.

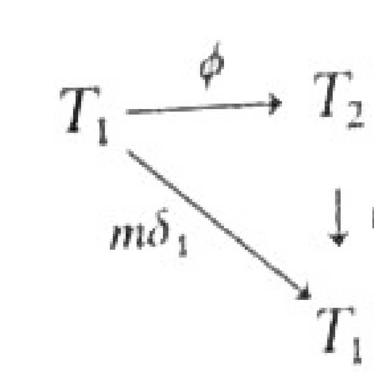
EXAMPLE. Let δ be the identity map on T, a complex torus. Let $m > 0$ be an integer. The map $m\delta: T \to T$ is an isogeny of degree m^{2d} where $d = \dim T$. To see this, write $T = V/\Lambda$. Then $\ker(m\delta) = (1/m)\Lambda/\Lambda \approx \Lambda/m\Lambda \approx (\mathbb{Z}/m\mathbb{Z})^{2d}$.

If $\phi_1: T_1 \to T_2$, and $\phi_2: T_2 \to T_3$ are isogenies, then so is $\phi_2 \circ \phi_1$ and degrees multiply: $\deg(\phi_2 \circ \phi_1) = \deg(\phi_1)\deg(\phi_2)$.

We say that T_1 and T_2 are *isogenous*, $T_1 \sim T_2$, if there is an isogeny ϕ: $T_1 \to T_2$. The next proposition shows that isogeny is an equivalence relation.

Proposition. *Let $d = \dim T_1 = \dim T_2$. If $\phi: T_1 \to T_2$ is an isogeny of degree m there is a unique isogeny $\psi: T_2 \to T_1$ of degree m^{2d-1} such that $\psi \circ \phi = m\delta_1$ and $\phi \circ \psi = m\delta_2$. ψ is called the* dual isogeny *to ϕ.*

PROOF. Since $\ker \phi \subseteq \ker(m\delta_1)$, a map ψ exists which makes the following diagram commutative:



One checks that ψ is the desired isogeny. Uniqueness is straightforward.

Since $\psi \circ \phi = m\delta_1$ one sees $(\phi \circ \psi - m\delta_2) \circ \phi = 0$. It follows that $\phi \circ \psi = m\delta_2$ because ϕ is onto. Finally, taking the degree of both sides of $\psi \circ \phi = m\delta_1$ yields $\deg \psi = m^{2d-1}$. □

Let the dual isogeny ψ be denoted by $\tilde{\phi}$. Then $\tilde{\tilde{\phi}} = m^{2d-2}\phi$ and $\widetilde{\phi_2 \circ \phi_1} = \tilde{\phi}_1 \circ \tilde{\phi}_2$.

We now begin our study of $\mathrm{End}(T)$. *We define* $\mathrm{End}_0(T) = \mathrm{End}(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. $\mathrm{End}_0(T)$ is a finite-dimensional $\mathbb{Q}$-algebra and $\mathrm{End}(T)$ can be considered as an order in it. We note that $\phi \in \mathrm{End}(T)$ is an isogeny if and only if it is invertible in $\mathrm{End}_0(T)$.

A complex torus T is called *simple* if it contains no proper complex subtorus. We say T is of *semisimple type* (this is not standard terminology) if it is isogenous to a product of simple complex tori. If T is simple then the usual Schur's lemma argument shows that $\mathrm{End}_0(T)$ is a division algebra.

Proposition. *If T is of semisimple type, then $\mathrm{End}_0(T)$ is a semisimple $\mathbb{Q}$-algebra.*

PROOF. Write $T \sim T_1^{n_1} \times T_2^{n_2} \times \cdots \times T_m^{n_m}$ where the T_j are simple and pairwise non-isogenous. Then, $\mathrm{End}_0(T) \approx M_{n_1}(D_1) \oplus \cdots \oplus M_{n_m}(D_m)$ where $D_j = \mathrm{End}_0(T_j)$ is a finite-dimensional division algebra over $\mathbb{Q}$. □

To study $\mathrm{End}_0(T)$ further we recall the complex representation

$$\rho_{\mathbb{C}}: \mathrm{End}_0(T) \to \mathrm{End}_{\mathbb{C}} V$$

and the rational representation $\rho_{\mathbb{Q}} = \rho_{\mathbb{Z}} \otimes \mathbb{Q}$

$$\rho_{\mathbb{Q}}: \mathrm{End}_0(T) \to \mathrm{End}_{\mathbb{Q}}(\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}).$$

Proposition. $\rho_{\mathbb{Q}} \otimes \mathbb{C} \approx \rho_{\mathbb{C}} \otimes \bar{\rho}_{\mathbb{C}}$.

See [SW, Lemma 39, p. 70], for the (simple) proof.

Let $\phi \in \mathrm{End}(T)$. We define the *characteristic polynomial*, $\mathrm{char}(\phi, x)$ *of* ϕ to be $\det(\rho_{\mathbb{Z}}(\phi) - xI)$. Note that $\mathrm{char}(\phi, x) \in \mathbb{Z}[x]$ is a monic polynomial of degree $2d$ where $d = \dim T$. We can easily extend this definition to $\phi \in \mathrm{End}_0(T)$.

Lemma. *Let $\psi \in \mathrm{End}(T)$. Then* $\det(\rho_{\mathbb{Z}}(\psi)) = \deg(\psi)$ (*if ψ is not an isogeny we define* $\deg(\psi) = 0$).

PROOF. $\rho_{\mathbb{Z}}(\psi): \Lambda \to \Lambda$ is 1–1 if and only if $\det(\rho_{\mathbb{Z}}(\psi)) \neq 0$. Thus, if ψ is an isogeny

$$\det(\rho_{\mathbb{Z}}(\psi)) = [\Lambda : \rho_{\mathbb{Z}}(\psi)\Lambda] = [\rho_{\mathbb{Z}}(\psi)^{-1}\Lambda : \Lambda] = \deg(\psi).$$ □

Using the lemma we can give an intrinsic characterization of $\mathrm{char}(\phi, x)$.

Proposition. *For all but finitely many integers n, $\phi - n\delta$ is an isogeny. The characteristic polynomial* $\mathrm{char}(\phi, x)$ *is the unique polynomial such that for all* $n \in \mathbb{Z}$, $\mathrm{char}(\phi, n) = \deg(\phi - n\delta)$.

PROOF. $\phi - n\delta$ is an isogeny if and only if $\det(\rho_{\mathbb{Z}}(\phi) - n\delta) \neq 0$. Thus if n is not a root of $\mathrm{char}(\phi, x)$, $\phi - n\delta$ is an isogeny. In this case, by the lemma, $\mathrm{char}(\phi, n) = \det(\rho_{\mathbb{Z}}(\phi) - n\delta) = \deg(\phi - n\delta)$. □

We note that this proposition makes sense in characteristic p and can be used to define the characteristic polynomial in the abstract theory.

We conclude this section by making a few remarks about the *l-adic representations*. For $n \in \mathbb{Z}$ define $T[n] = \ker(n\delta)$. As we have seen, $T[n]$ is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^{2d}$. If $\phi \in \mathrm{End}(T)$ then $\phi(T[n]) \subseteq T[n]$. For a prime number l consider the inverse system $\{T[l^m] \mid m \geq 1\}$ where $l\delta: T[l^{m+1}] \to T[l^m]$ are the transition maps. An endomorphism ϕ induces a map of this inverse system and thus acts on $\mathrm{proj\,lim}\, T[l^m] = T_l(T)$, the *l-adic Tate module*. Let $V_l(T) = T_l(T) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$. Then we have a representation

$$\rho_{\mathbb{Q}_l}: \mathrm{End}_0(T) \to \mathrm{End}_{\mathbb{Q}_l}(V_l(T)).$$

It is easy to check that $T_l(T) \approx \Lambda \otimes \mathbb{Z}_l$ both as a $\mathbb{Z}_l$ and as an $\mathrm{End}(T)$ module. Thus the l-adic representations are all equivalent to the rational representation. In working over $\mathbb{C}$ they provide no new information. However, when working with abelian varieties over arbitrary fields the l-adic representations can always be defined whereas an analogue of the rational representation need not exist.

§3. Abelian Varieties

We will be using some standard terminology from the theory of complex analytic manifolds. We assume known the definitions of holomorphic and meromorphic functions on such manifolds, as well as the definitions of divisors, positive divisors, etc. See [SW, §3], for a concise discussion. Another good reference is [SHAF, Chap. VIII].

Let $\mathcal{M}(T)$ be the field of meromorphic functions on the complex torus T. Since T is compact the only holomorphic functions are constants. How big is $\mathcal{M}(T)$? How can one construct elements of $\mathcal{M}(T)$? We quote a general theorem of Siegel. See [SHAF] for a proof.

Theorem. *Let M be a compact, connected, complex manifold of dimension d. Then $\mathcal{M}(M)$ has transcendence degree over $\mathbb{C}$ at most d. If d is attained then $\mathcal{M}(M)$ is a finitely generated field over $\mathbb{C}$.*

If $M = X(\mathbb{C})$, the complex points on a non-singular algebraic variety X, then $\mathcal{M}(M) \approx \mathbb{C}(X)$, the field of rational functions on X. Thus, in this case,

$\mathcal{M}(M)$ is a finitely generated field of transcendence degree $d = \dim X$. This shows that if the complex torus T is an abelian variety $\mathcal{M}(T)$ is "big." Later on we will give an example of a torus T with $\mathcal{M}(T) = \mathbb{C}$. This never happens when T has dimension 1. Let $E = \mathbb{C}/\Lambda$ be a one-dimensional complex torus. Then $\mathcal{M}(E)$ is generated by the Weierstrass elliptic function $\mathcal{P}(z, \Lambda)$ and its derivative $\mathcal{P}'(z, \Lambda)$. These functions are connected by the well-known equation

$$\mathcal{P}'(z, \Lambda)^2 = 4\mathcal{P}(z, \Lambda)^3 - g_2\mathcal{P}(z, \Lambda) - g_3,$$

where g_2 and g_3 are constants satisfying $g_2^3 - 27g_3^2 \neq 0$. The map

$$E \to [1, \mathcal{P}(z, \Lambda), \mathcal{P}'(z, \Lambda)]$$

extends to an imbedding of E into $\mathbb{P}^2$ as a non-singular cubic plane curve. Thus one-dimensional complex tori are one-dimensional abelian varieties or elliptic curves.

In higher dimensions the situation is more complicated. There are non-trivial conditions on a complex torus in order that it correspond to an abelian variety. To explain these conditions we need to review some linear algebra.

Suppose V is a finite-dimensional complex vector space. A map

$$H: V \times V \to \mathbb{C}$$

is a *Hermitian form on* V if:

(i) for fixed $v \in V$,

$$u \mapsto H(u, v)$$

is a linear map $V \to \mathbb{C}$;

(ii) for fixed $u \in V$,

$$v \mapsto H(u, v)$$

is an *antilinear* map $V \to \mathbb{C}$;

(iii) $H(u, v) = \overline{H(v, u)}$ for all u, v in V.

(Of course, (i) + (iii) $\Rightarrow$ (ii)).

If H is a Hermitian form, then we will always write S for the real part of H and E for the imaginary part.

Thus $H(u, v) = S(u, v) + iE(u, v)$, $u, v \in V$, and $S, E: V \times V \to \mathbb{R}$ are real bilinear. We also see that:

$$S(u, v) = E(iu, v),$$

$$S(iu, iv) = S(u, v), \qquad E(iu, iv) = E(u, v),$$

S *is symmetric* $\qquad (S(u, v) = S(v, u))$,

E is antisymmetric $\qquad (E(u, v) = -E(v, u))$.

Conversely, if E is a real, antisymmetric bilinear form on V satisfying

$E(iu, iv) = E(u, v)$, then $H(u, v) = E(iu, v) + iE(u, v)$ is a Hermitian form. The set of Hermitian forms on V form a group under pointwise addition and subtraction.

Definition. Suppose $T = V/\Lambda$ is a complex torus. A *Riemann form* on T is a Hermitian form H on V such that $E = \operatorname{Im} H$ is integer valued on Λ, i.e. $E(\lambda_1, \lambda_2) \in \mathbb{Z}$, for all $\lambda_1, \lambda_2 \in \Lambda$. If $H(u, u) \geq 0$ for all $u \in V$ we say H is a *positive Riemann form*. If H is positive definite, i.e. $H(u, u) > 0$ for all $u \in V$, $u \neq 0$, we say H is a *non-degenerate Riemann form* on T.

One sometimes calls H a *Hermitian Riemann form* on T, and $E = \operatorname{Im} H$ an *alternating Riemann form* on T.

Theorem A. *A complex torus T is the manifold of complex points on an abelian variety if and only if T possesses a non-degenerate Riemann form.*

The proof will be sketched later. The idea is to construct *theta functions* using the non-degenerate Riemann form on T and use these to construct a projective embedding of T.

We will now discuss some naturally occurring Riemann forms. If $\dim T = 1$ we have $T = \mathbb{C}/\Lambda$ where $\Lambda = \mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2$ with $\operatorname{Im}(\lambda_1/\lambda_2) > 0$. Regard $\mathbb{C}$ as a two-dimensional vector space over $\mathbb{R}$, and define $E(z, w)$ by the equation $z \wedge w = E(z, w)\lambda_1 \wedge \lambda_2$. Then $E(z, w)$ is a Riemann form on T. $E(iz, iw) = E(z, w)$ follows from the fact that multiplication by i is area preserving. Every other Riemann form on T is an integral multiple of E. Thus, the Riemann form does not usually occur explicitly in the theory of elliptic functions.

Here is another class of complex tori for which it is possible to explicitly write down a Riemann form. Suppose K is a CM field, i.e. a totally imaginary quadratic extension of a totally real number field, K^+. Examples are provided by imaginary quadratic number fields, and cyclotomic fields. Set $[K:\mathbb{Q}] = 2d$, and let $\Phi = \{\phi_1, \ldots, \phi_d\}$ be a subset of distinct complex imbeddings $K \subsetneq \mathbb{C}$ such that if $\phi \in \Phi$, $\bar{\phi} \notin \Phi$, where $\bar{\phi}$ is the complex conjugate embedding. Φ provides an isomorphism, which we continue to call Φ, of $K \otimes_{\mathbb{Q}} \mathbb{R}$ with $\mathbb{C}^d$, which takes $\alpha \otimes 1$ to $(\phi_1(\alpha), \phi_2(\alpha), \ldots, \phi_d(\alpha))$. Let $\mathcal{A}$ be an integral ideal in K. It can be shown that $\Phi(\mathcal{A})$ is a lattice in $\mathbb{C}^d$. Set $A = \mathbb{C}^d/\Phi(\mathcal{A})$. We proceed to find a Riemann form on A. A simple calculation shows we can find an algebraic integer $\xi \in K$ such that $K = K^+(\xi)$, $-\xi^2 \in K^+$ and is totally positive, and $\operatorname{Im} \phi_j(\xi) > 0$ for $j = 1, \ldots, d$. For $z, w \in \mathbb{C}^d$ define $E(z, w) = \sum_{j=1}^{d} \phi_j(\xi)(\bar{z}_j w_j - z_j \bar{w}_j)$. $E(z, w)$ is $\mathbb{R}$-bilinear, anti-symmetric, and $E(iz, w)$ is symmetric and positive definite. A calculation shows that for $\alpha, \beta \in K$, $E(\Phi(\alpha), \Phi(\beta)) = t(\xi\tilde{\alpha}\beta)$ where $\alpha \to \tilde{\alpha}$ is the non-trivial automorphism of K/K^+ and t is the trace from K to $\mathbb{Q}$. Thus, $E(z, w)$ takes integral values on $\Phi(\mathcal{A})$, and is a non-degenerate Riemann form on A. The ring of integers $\mathcal{O}_K$ of K imbeds in $\operatorname{End}(\mathcal{A})$ via the map which associates to ω the diagonal matrix whose iith coefficient is $\phi_i(\omega)$. Thus, $K \subsetneq \operatorname{End}_0(A)$. In this situation, A is said to admit complex multiplication by K and the corresponding abelian variety is said to be of CM type (K, Φ). The assumption that $\mathscr{A}$ is an integral ideal is unnecessarily restrictive. It suffices to assume that $\mathscr{A}$ is a $\mathbb{Z}$-lattice in K. For this and much more on abelian varieties of CM type see [L–CM] and [SHIM].

Returning to the general theory we make the following convenient definition. A complex torus is an *abelian manifold* if it possesses a non-degenerate Riemann form.

Restricting the Riemann form shows that a subtorus of an abelian manifold is again an abelian manifold. One can show that a quotient of an abelian manifold is also an abelian manifold. This is a corollary of the following important result.

Theorem (Poincaré Reducibility Theorem). *Suppose A is an abelian manifold and $A_1 \subset A$ an abelian submanifold. Then there is an abelian submanifold A_2 such that $A_1 \cap A_2$ is finite and A is isogenous to $A_1 \times A_2$.*

PROOF. *Sketch.* Write $A = V/\Lambda$ with Riemann form H. Then $A_1 = V_1/\Lambda_1$ where $V_1 \subseteq V$ is a complex subspace and $\Lambda_1 = V_1 \cap \Lambda$. Set $V_2 = V_1^\perp$, the orthogonal complement of V_1 with respect to H, and set $\Lambda_2 = V_2 \cap \Lambda$. It can be shown that Λ_2 is a lattice in V_2 and so $A_2 = V_2/\Lambda_2$ is an abelian submanifold of A. Moreover, $\Lambda_1 + \Lambda_2$ is of finite index in Λ. The map $A_1 \times A_2 \to A$ given by $(a_1, a_2) \to a_1 + a_2$ is an isogeny. See [SW, Theorem 34, Cor. 3] or [L–A, p. 117] for more details. □

Corollary. *An abelian manifold A is of semisimple type and so $\mathrm{End}_0(A)$ is a semisimple $\mathbb{Q}$-algebra (see the second proposition in Section 2).*

We will now discuss some analytic results which will lead to the introduction of theta functions. We will explain Poincaré's basic result (Theorem B) that every periodic divisor is generated by a theta function, and then the important theorem of Frobenius (Theorem C) which computes the dimension of a certain vector space of theta functions. Then, finally, we will be in a position to state the Lefschetz Embedding Theorem (Theorem D) of which Theorem A is an immediate consequence.

Suppose V is a d-dimensional vector space over $\mathbb{C}$ and Λ is a lattice in V. Put $T = V/\Lambda$ and let $\pi: V \to T$ be the projection map. A function f on V is *periodic* with respect to Λ if $f(z + \lambda) = f(z)$ for all $z \in V$, $\lambda \in \Lambda$. Such a function gives rise to a function on T, and conversely if g is a function on T then $f = g \circ \pi$ is a periodic function on V. A *Cartier divisor* D on V is given by a family $\{(U_\alpha, f_\alpha)\}$ where the U_α form an open covering of V, f_α is meromorphic on U_α, not identically zero, and f_α/f_β is holomorphic on $U_\alpha \cap U_\beta$ for all α, β. The divisor is called *positive* if the function f_α are holomorphic. If $a \in V$ the *translate* of D by a, D_a, is given by $\{(U_\alpha + a, f_\alpha(z - a)\}$. If $D_\lambda = D$ for all $\lambda \in \Lambda$ we say D is a *periodic divisor*. Note that the divisor of a periodic meromorphic function is a periodic divisor. Let $\mathscr{D}(V)$ and $\mathscr{D}(T)$ be the group of

divisors on V and T respectively. Then π induces a homomorphism

$$\pi^*: \mathscr{D}(T) \to \mathscr{D}(V).$$

The image of π^* consists of the periodic divisors.

Divisors on V are easier to analyze than those on T. For example, we have the following facts:

(1) Suppose $g \in \mathscr{M}(V)$ has trivial divisor. Then g is a nowhere vanishing holomorphic function and we can write

$$g(z) = \mathbf{e}(h(z)),$$

where $h(z)$ is holomorphic and $\mathbf{e}(z) = \exp(2\pi i z)$, $i = \sqrt{-1}$.

(2) (Cousin's Theorem). Every divisor on V is principal, i.e. we can set all $f_\alpha = f$, a single function meromorphic on V. In cohomological terms this says $H^1(V, \mathcal{O}^*) = (0)$ where $\mathcal{O}^*$ is the sheaf of nowhere vanishing holomorphic functions on V.

Suppose D' is a divisor on T and $\pi^*(D') = D$ is the corresponding periodic divisor on V. By Cousin's theorem, $D = (f)$. Since $D_\lambda = D$ for all $\lambda \in \Lambda$, we see $f(z + \lambda) = U_\lambda(z) f(z)$ for all $\lambda \in \Lambda$ where $U_\lambda(z)$ is a nowhere vanishing holomorphic function. Thus, by (1), $U_\lambda(z) = \mathbf{e}(h_\lambda(z))$ where $h_\lambda(z)$ is holomorphic. Setting $\lambda = \lambda_1 + \lambda_2$ we find the following consistency condition

$$h_{\lambda_1+\lambda_2}(z) \equiv h_{\lambda_1}(z + \lambda_2) + h_{\lambda_2}(z) \mod \mathbb{Z}.$$

We wish to choose $h_\lambda(z)$ to be as simple as possible. The simplest choice leading to a fruitful theory is

$$h_\lambda(z) = L(z, \lambda) + J(\lambda)$$

where $L(z, \lambda)$ is linear in z, and $J(\lambda)$ is a constant.

Definition. Let $L: V \times \Lambda \to \mathbb{C}$ and $J: \Lambda \to \mathbb{C}$ be maps with $L(z, \lambda)$ linear in z for all $\lambda \in \Lambda$. A holomorphic (resp. meromorphic) *theta function* for Λ of *type* (L, J) is a holomorphic (resp. meromorphic) function θ on V such that $\theta(z + \lambda) = \mathbf{e}(L(z, \lambda) + J(\lambda))\theta(z)$ for all $z \in V$, $\lambda \in \Lambda$.

Theorem B (Poincaré). *For every divisor D' on T, the periodic divisor $\pi^*(D') = D$ is the divisor of a meromorphic theta function, $\pi^*(D') = (\theta)$. If D' is a positive (holomorphic) divisor, then θ is a holomorphic theta function.*

Note that Theorem B is a sharp form of Cousin's theorem in the special case of periodic divisors. The early proofs were quite complicated. The proof usually quoted today is due to A. Weil ("Theorémes fondamentaux de la théorie des fonctions thêta", *Seminar Bourbaki*, 1948/49).

A question which naturally arises is to what extent θ is determined by D'. An easy exercise shows that a theta function has trivial divisor if and only if $\theta(z) = \mathbf{e}(q(z) + l(z) + c)$ where $q(z)$ is a quadratic form, $l(z)$ is linear, and c is a constant. We call such theta functions *trivial theta functions*. Then:

$$\mathscr{D}(T) \approx \frac{\text{group of theta functions}}{\text{trivial theta functions}}.$$

Suppose θ is a theta funcction of type (L, J). We now show how to associate a Hermitian form to θ.

The consistency conditions explained earlier impose the following restrictions on L and J:

(a) $J(\lambda + \mu) - J(\lambda) - J(\mu) \equiv L(\lambda, \mu) \mod \mathbb{Z}$;
(b) $L(\lambda, \mu) \equiv L(\mu, \lambda) \mod \mathbb{Z}$;
(c) $L(z, \lambda + \mu) = L(z, \lambda) + L(z, \mu)$.

Note that (b) follows from (a). Condition (c) implies that $L(z, \lambda)$ can be extended to an $\mathbb{R}$-bilinear function on V. Define $E(z, w) = L(z, w) - L(w, z)$. Then, E is an anti-symmetric $\mathbb{R}$-bilinear function on $V \times V$ which assumes integer values on $\Lambda \times \Lambda$, by (b). This last condition implies E is real valued on $V \times V$. Moreover, we have the following result.

Lemma. $E(iz, iw) = E(z, w)$.

PROOF. $E(iz, iw) = L(iz, iw) - L(iw, iz) = i(L(z, iw) - L(w, iz))$ and $E(z, w) = L(z, w) - L(w, z) = -i(L(iz, w) - L(iw, z))$. Thus $E(iz, iw) - E(z, w) = i(E(iz, w) - E(iw, z))$ must be zero since it is in $\mathbb{R} \cap i\mathbb{R}$. □

Define $H(z, w) = E(iz, w) + iE(z, w)$. Then H is a Riemann form on $T = V/\Lambda$, called *the Riemann form associated to* θ.

Suppose $\theta(z) = \mathbf{e}(q(z) + l(z) + c)$ is a trivial theta function. Let $B(z, w) = q(z + w) - q(z) - q(w)$. $B(z, w)$ is $\mathbb{C}$-bilinear and symmetric. A short calculation shows that $\theta(z + \lambda) = \mathbf{e}(B(z, \lambda) + q(\lambda) + l(\lambda))\theta(z)$. Thus $E(z, \lambda) = B(z, \lambda) - B(\lambda, z) = 0$ and the Riemann form associated with a trivial theta function is zero. Thus there is a homomorphism from the divisor group $\mathscr{D}(T)$ to the group of Hermitian forms on T given by:

$$D' \to \pi^*(D') = (\theta) \to H.$$

We can refine this further.

Proposition. *Suppose $D = \pi^*(D')$ is a positive divisor. If $D = (\theta)$, then θ is an entire function and the corresponding Riemann form H is positive, i.e. $H(z, z) \geq 0$ for all $z \in V$.*

See [SW, p. 31, Lemma 31] for the proof. If H is positive definite we say θ is *non-degenerate* and that the corresponding divisor on T is *ample*. That ample divisors in this sense are ample in the sense of algebraic geometry will be shown later.

Since we can multiply a given theta function by a trivial theta function without changing the corresponding divisor it is natural to look for a normal form.

Proposition. *Let θ be a theta function and H the associated Riemann form. Then there is a theta function $\tilde{\theta}$, unique up to a multiplicative constant, such that $\tilde{\theta}/\theta$ is a trivial theta function, and*

$$\tilde{\theta}(z + \lambda) = \mathbf{e}\left(\frac{1}{2i} H\left(z + \frac{\lambda}{2}, \lambda\right) + K(\lambda)\right)\tilde{\theta}(z),$$

where $K(\lambda)$ is real valued and

$$K(\lambda_1 + \lambda_2) - K(\lambda_1) - K(\lambda_2) \equiv \tfrac{1}{2}E(\lambda_1, \lambda_2) \mod \mathbb{Z}.$$

$\tilde{\theta}$ is called the *normalized theta function* associated to θ. Set $\psi(\lambda) = \mathbf{e}(K(\lambda))$. Then ψ satisfies $\psi(\lambda_1 + \lambda_2) = \psi(\lambda_1)\psi(\lambda_2)\mathbf{e}(\frac{1}{2}E(\lambda_1, \lambda_2))$. ψ is called *the associated quadratic character* of θ. Note that $|\psi(\lambda)| = 1$.

If $\mathrm{Th}(L, J)$ denotes the vector space of theta functions of type (L, J) and $\mathrm{Th}_{\mathrm{norm}}(H, \psi)$ the space of normalized theta functions with associated Riemann form H and quadratic character ψ, then one can find a trivial theta function θ_0 such that multiplication by θ_0 gives an isomorphism

$$\mathrm{Th}(L, J) \approx \mathrm{Th}_{\mathrm{norm}}(H, \psi).$$

See [L–A, Chap. VI, §2] for this and the proof of the above proposition.

At this point it is easy to show that every $f \in \mathscr{M}(T)$ can be represented as a quotient of holomorphic theta functions of the same type. One can write $(f) = D_0 - D_\infty$ where D_0 and D_∞ are positive divisors. $\pi^*(D_\infty) = (\theta_\infty)$ by Poincaré's theorem, Theorem B. Now, $\pi^*(f)\theta_\infty$ has divisor $\pi^*(D_0)$ and so is a holomorphic theta function, θ_0, of the same type as θ_∞. Thus $\pi^*f = \theta_0/\theta_\infty$ as asserted. Conversely, the quotient of two theta functions of the same type is a periodic meromorphic function. This leads to the problem of constructing all holomorphic theta functions of a given type. This is accomplished by a theorem of Frobenius. Before stating this theorem it is necessary to review the definition of the Pfaffian of an alternating form.

Let x_{ij}, $1 \leq i < j \leq 2d$ be $d(2d - 1)$ elements algebraically independent over $\mathbb{Q}$. Set $x_{ji} = -x_{ij}$ and consider the antisymmetric matrix $X = (x_{ij})$. There is a unique polynomial $\mathrm{Pf}(x)$ of degree d such that $\det X = \mathrm{Pf}(x)^2$ and $\mathrm{Pf}(x)$ takes the value 1 on

$$\begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix}.$$

If $G = (g_{ij})$ is an antisymmetric $2d \times 2d$ matrix with coefficients in any ring we define $\mathrm{Pf}(G)$ to be the value of $\mathrm{Pf}(x)$ when g_{ij} is substituted for x_{ij}. $\mathrm{Pf}(G)$ is called the *Pfaffian* of G. Let Λ be a free $\mathbb{Z}$-module of rank $2d$ and E an alternating form on Λ. If $\{\lambda_1, \lambda_2, \ldots, \lambda_{2d}\}$ is a basis of Λ, set $\mathrm{Pf}(E) = \mathrm{Pf}(E(\lambda_i, \lambda_j))$. This is well defined up to sign.

Lemma. *Let Λ be a free $\mathbb{Z}$-module of rank 2d and E a non-degenerate alternating form on Λ. Then there is a basis $\{\lambda_1, \lambda_2, \ldots, \lambda_{2d}\}$ of Λ such that $E(\lambda_i, \lambda_j) = 0$ for $1 \le i, j \le d$, $E(\lambda_{d+i}, \lambda_{d+j}) = 0$ for $1 \le i, j \le d$, and $E(\lambda_i, \lambda_{d+j}) = e_i\delta_{ij}$ for $1 \le i, j \le d$ where $e_1|e_2|\ldots|e_d$ are positive integers. Finally, $\mathrm{Pf}(E) = e_1e_2\ldots e_d$.*

This lemma is due to Frobenius. A basis with the given properties is called a *symplectic basis* for Λ. If $\mathscr{E}$ is the diagonal matrix with diagonal entries $e_1, e_2, \ldots, e_d$ then $(E(\lambda_i, \lambda_j))$ has the form

$$\begin{pmatrix} 0 & \mathscr{E} \\ -\mathscr{E} & 0 \end{pmatrix}.$$

Theorem C (Frobenius). *Suppose (L, J) is a type, H the associated Riemann form, E = Im H. Assume H is positive definite. The vector space of holomorphic theta functions of type (L, J) over $\mathbb{C}$ has dimension Pf(E).*

PROOF. *Sketch.* Choose a symplectic basis for Λ with respect to E. It is easy to check that $\lambda_1, \lambda_2, \ldots, \lambda_d$ are a basis for V over $\mathbb{C}$. Let $z_1, z_2, \ldots, z_d$ be the coordinate functions on V with respect to this basis. By multiplying by a suitable trivial theta function, the space that we are examining is isomorphic to the space of holomorphic θ on V satisfying the equations

$$\theta(z + \lambda_i) = \theta(z), \qquad 1 \le i \le d,$$

$$\theta(z + \lambda_{d+i}) = \mathbf{e}(e_i z + c_i)\theta(z), \qquad 1 \le i \le d,$$

for some fixed constants $c_1, c_2, \ldots, c_d$. The first set of these equations show that we can expand θ as a Fourier series

$$\theta(z) = \sum_{n \in \mathbb{Z}^d} a(n)\mathbf{e}(n \cdot z).$$

The second set of equations imposes recurrence relations on the set of coefficients a(n) which show that all the a(n) can be expressed in terms of those a(n) with $0 \le n_i \le e_i - 1$ where $n = (n_1, n_2, \ldots, n_d)$. This gives an upper bound of $e_1e_2\ldots e_d = \mathrm{Pf}(E)$ on the dimension of the given space of theta functions. To get equality one must show that for a collection of a(n) satisfying the recurrence relations the corresponding formal Fourier series is in fact a holomorphic function. We omit the proof but note that here the assumption that H is positive definite comes into play. □

Theorems B and C allow us to answer all the questions previously raised about complex tori. We first discuss a form of the Riemann–Roch theorem.

Let D be a positive divisor on T and define, as usual

$$\mathscr{L}(D) = \{f \in \mathscr{M}(T) | (f) + D \ge 0\}.$$

There is a holomorphic theta function θ_0 such that $\pi^*(D) = (\theta_0)$. Define $\mathscr{L}(\theta_0)$ to be the space of all holomorphic theta functions with the same type as θ_0. Then $\theta \to \theta/\theta_0$ gives an isomorphism of $\mathscr{L}(\theta_0)$ with $\mathscr{L}(D)$. Theorem C gives the dimension of this space.

Theorem. *Suppose $D_0, D_1, \ldots, D_m$ are positive divisors on T and that D_0 is ample. Then there is a polynomial P of degree d such that*

$$\dim_{\mathbb{C}} \mathscr{L}\left(\sum_{j=0}^{m} r_j D_j\right) = P(r_0, r_1, \ldots, r_m)$$

whenever $r_j \ge 0$ for all j and $r_0 > 0$.

PROOF. Set $\pi^*(D_j) = (\theta_j)$ and let H_j be the Hermitian form corresponding to θ_j. Then $\sum_{j=0}^m r_j H_j$ corresponds to $\prod_{j=0}^m \theta_j^{r_j}$ and is positive definite if $r_j \ge 0$ for $1 \le j \le m$ and $r_0 > 0$. Then we have

$$\dim_{\mathbb{C}} \mathscr{L}\left(\sum_{j=0}^{m} r_j D_j\right) = \dim_{\mathbb{C}} \mathscr{L}(\theta_0^{r_0}\theta_1^{r_1}\ldots\theta_m^{r_m})$$

$$= \mathrm{Pf}(r_0E_0 + r_1E_1 + \cdots + r_mE_m)$$

which is a polynomial in the r_j of the type described in the theorem. □

Corollary. *If D is an ample divisor on T, then $\dim_{\mathbb{C}} \mathscr{L}(rD) = r^d \dim_{\mathbb{C}} \mathscr{L}(D)$ for $r > 0$.*

Suppose T is a complex torus of dimension d which possesses an ample divisor. The above theorem can be used to prove Siegel's theorem (see the beginning of this section) for T. We prove the first part as follows. Suppose $f_1, f_2, \ldots, f_m \in \mathscr{M}(T)$ with $m > d$, where $d = \dim T$. There exists an ample divisor D such that $(f_j) + D \ge 0$ for $1 \le j \le m$. Set $\pi^*(D) = (\theta_0)$ and $\theta_j = f_j\theta_0$. The θ_j are holomorphic theta functions of the same type as θ_0. For $r = \sum_{j=1}^m r_j$ there are $\binom{m+r}{r}$ monomials $\theta_0^{r_0}\theta_1^{r_1}\ldots\theta_m^{r_m}$ as the r_j vary over non-negative integers. These are all in $\mathscr{L}(\theta_0^r)$ which has dimension $r^d\mathrm{Pf}(E)$ where E is the alternating form corresponding to θ_0. Since $m > d$ we have $\binom{m+r}{m} > r^d\mathrm{Pf}(E)$ for large r. The corresponding monomials are then linearly dependent and this gives an algebraic relation among the f_j. Thus the transcendence degree of $\mathscr{M}(T)$ over $\mathbb{C}$ is $\le d$. See [L–A, Chap. VI, §6] for the proof that when equality holds, $\mathscr{M}(T)$ is finitely generated over $\mathbb{C}$.

Let A be an abelian manifold. Recall that this means A is a complex torus with a non-degenerate Riemann form H, i.e. H is positive definite. Frobenius' theorem, Theorem C, implies the existence of a theta function θ on A with H its corresponding Hermitian form. Let D be the divisor on A induced by θ.

Theorem D (Lefschetz Embedding Theorem). *Let A be an abelian manifold and D the divisor on A constructed in the above remarks. Then $\mathscr{L}(3D)$ considered as a linear system on A gives a projective embedding $A \to \mathbb{P}^N(\mathbb{C})$.*

See [L–A] or [SW] for the proof. The map is obtained as follows. $\mathscr{L}(3D) \approx \mathscr{L}(\theta^3)$. Let $\theta_0, \theta_1, \theta_2, \ldots, \theta_N$ be a basis of $\mathscr{L}(\theta^3)$. Then $t \in A$ goes to $[\theta_0(t), \theta_1(t), \ldots, \theta_N(t)] \in \mathbb{P}^N(\mathbb{C})$. Since all the θ_j are of the same type, the map is well defined. One must show it is defined everywhere, is 1–1, and that the image is a non-singular subvariety of $\mathbb{P}^N(\mathbb{C})$.

Let $E = \operatorname{Im} H$ be the alternating Riemann form corresponding to H. Then $N = \dim \mathscr{L}(\theta^3) - 1 = 3^d \operatorname{Pf}(E) - 1$ where $d = \dim_{\mathbb{C}} A$. One can also show that the degree of the embedding is $d!\, 3^d \operatorname{Pf}(E)$.

Suppose $d = 1$. Then $N = 2$ and the degree of the embedding is 3 if we use the Riemann form constructed earlier on $\mathbb{C}/\Lambda = A$; A embeds as a non-singular plane cubic. Thus, the Lefschetz Embedding Theorem can be considered as a vast generalization of the work of Weierstrass on elliptic functions. We remark in passing that the theta function that arises in this context is the Weierstrass σ-function, $\sigma(z)$, whose induced divisor on $\mathbb{C}/\Lambda$ is just the zero element.

Theorem D is, of course, a very explicit form of Theorem A. Conversely, if A is an abelian variety over $\mathbb{C}$ then A has a projective embedding. The pull-back of a hyperplane section is an ample divisor D on A and if $\pi^*(D) = (\theta)$ then the Hermitian form corresponding to θ is a non-degenerate Riemann form on $A(\mathbb{C})$, i.e. $A(\mathbb{C})$ is an abelian manifold. Thus the existence of a non-degenerate Riemann form is a necessary and sufficient condition for a complex torus to be the manifold of complex points on an abelian variety!

§4. The Néron–Severi Group and the Picard Group

In this section we assume $A = V/\Lambda$ is an abelian manifold. We define some groups of divisors on A.

$\mathscr{D}$ = group of all divisors on A.

$\mathscr{D}_a$ = group of divisors on A whose corresponding Riemann form is 0.

$\mathscr{D}_l$ = group of principal divisors.

The divisors in $\mathscr{D}_a$ are said to be *algebraically equivalent to zero*, those in $\mathscr{D}_l$ are said to be *linearly equivalent to zero*. We have $\mathscr{D}_l \subseteq \mathscr{D}_a \subseteq \mathscr{D}$. Define the *Néron–Severi group* to be $\operatorname{NS}(A) = \mathscr{D}/\mathscr{D}_a$, the *Picard Group* to be $\operatorname{Pic}(A) = \mathscr{D}/\mathscr{D}_l$, and $\operatorname{Pic}^0(A) = \mathscr{D}_a/\mathscr{D}_l$. Then we have the exact sequence

$$(0) \to \operatorname{Pic}^0(A) \to \operatorname{Pic}(A) \to \operatorname{NS}(A) \to (0).$$

Proposition. $\operatorname{NS}(A)$ *is a torsion free finitely generated abelian group (and so a free abelian group) of* rank $\leq d(2d-1)$ *where* $d = \dim A$.

PROOF. The Riemann form H associated to a divisor is completely determined by $E = \operatorname{Im} H$ restricted to Λ. These form a group isomorphic to a subgroup of the $2d \times 2d$ antisymmetric matrices with integer coefficients. The latter group is free abelian of rank $d(2d-1)$. Thus $\operatorname{NS}(A)$ injects into a free abelian group of rank $d(2d-1)$. □

When $d = 1$, $\operatorname{NS}(A) \approx \mathbb{Z}$, the isomorphism being given by $D \to \deg(D)$.

We will next show that $\operatorname{Pic}^0(A)$ can be given the structure of an abelian manifold, $\hat{A}$, *the dual abelian manifold of* A.

Suppose $[D]$ is the class in $\operatorname{Pic}^0(A)$ of a divisor D. Then D corresponds to a normalized theta function θ which only depends on the class of D. Since the Riemann form associated to θ is trivial we have $\theta(z + \lambda) = e(K(\lambda))\theta(z)$ where $K(\lambda) \in \mathbb{R}$ and satisfies $K(\lambda_1 + \lambda_2) \equiv K(\lambda_1) + K(\lambda_2) \bmod \mathbb{Z}$. Then $\chi_D(\lambda) = e(K(\lambda))$ is a character of Λ. Note that χ_D is the trivial character if and only if $\theta \in \mathscr{M}(A)$. Denoting the Pontryagin dual of Λ by $\hat{\Lambda}$ we get a monomorphism $\operatorname{Pic}^0(A) \subsetneq \hat{\Lambda}$ by $[D] \to \chi_D$. Since $\Lambda \approx \mathbb{Z}^{2d}$ we have $\hat{\Lambda} \approx (\mathbb{R}/\mathbb{Z})^{2d}$ is a real torus of dimension $2d$. To give $\operatorname{Pic}^0(A)$ the structure of an abelian manifold we will show $\operatorname{Pic}^0(A) \subsetneq \hat{\Lambda}$ is an isomorphism, that $\hat{\Lambda}$ has the structure of a *complex* torus, and finally that the resulting complex torus has a non-degenerate Riemann form.

Suppose X is an ample divisor on A, θ the corresponding theta function, H the corresponding Riemann form, and $E = \operatorname{Im} H$. For $t \in V$ we set $\theta_t(z) = \theta(z - t)$. The divisor corresponding to θ_t is X_t, the translation of X by t (actually, by the image of t in A). A calculation shows that the normalized theta function associated to θ_t/θ has multiplier $e(-E(t, \lambda))$. In the first place this shows $X_t - X$ is algebraically equivalent to zero. Secondly, since E is non-degenerate, every character of $\hat{\Lambda}$ is of the form $\lambda \to e(-E(t, \lambda))$ for suitable $t \in V$. This proves

Proposition. $\operatorname{Pic}^0(A) \to \hat{\Lambda}$ *given by* $D \to \chi_D$ *is an isomorphism. Moreover, every* $D \in \mathscr{D}_a$ *is linearly equivalent to* $X_t - X$ *for suitable* $t \in A$.

Corollary. *If* X *is an ample divisor on* A, *the map* $\phi_X\colon A \to \operatorname{Pic}^0(A)$ *given by* $\phi_X(t) = [X_t - X]$ *is surjective with finite kernel of order* $\det(E) = \operatorname{Pf}(E)^2$.

PROOF OF COROLLARY. The surjectivity is given by the theorem. The kernel is precisely $\{t \in V \mid E(t, \lambda) \in \mathbb{Z}$, for all $\lambda \in \Lambda\}/\Lambda$. It is straightforward to see this is a finite group of order $\operatorname{Pf}(E)^2$ (use a symplectic basis for Λ). □

To put the structure of a complex torus on $\operatorname{Pic}^0(A) \approx \hat{\Lambda}$ we consider the space V^* of antilinear functionals on V. Explicitly,

$$V^* = \{f \in \operatorname{Hom}_{\mathbb{R}}(V, \mathbb{C}) \mid f(\alpha t) = \bar{\alpha} f(t), \alpha \in \mathbb{C}, t \in V\}.$$

V^* is a complex vector space of the same dimension as V. We have a non-degenerate $\mathbb{R}$-bilinear pairing $\langle\ ,\ \rangle\colon V^* \times V \to \mathbb{R}$ given by $\langle \xi, t\rangle = \operatorname{Im} \xi(t)$.

Define Λ^* by $\Lambda^* = \{\xi \in V^* | \langle \xi, \lambda \rangle \in \mathbb{Z}, \text{ for all } \lambda \in \Lambda\}$. It is not hard to see that Λ^* is a lattice in V^*. The following lemma follows from the definitions and the non-degeneracy of the pairing $\langle \xi, \lambda \rangle$.

Lemma. *For* $\xi \in V^*$ *define* $\chi_\xi(\lambda) = \mathbf{e}(-\langle \xi, \lambda \rangle)$. *Then* $\xi \to \chi_\xi$ *gives an isomorphism from* V^*/Λ^* *to* $\hat{\Lambda}$.

This gives $\hat{\Lambda}$ the structure of a complex torus.

If H is the given Riemann form on A, then $t \to H(t, \cdot)$ is an isomorphism of V with V^* as complex vector spaces. Under this isomorphism one checks that Λ goes to Λ^*. Thus we have an epimorphism $\phi_H\colon V/\Lambda \to V^*/\Lambda^*$. The various maps we have defined are tied together by the following commutative diagram

$$\begin{array}{ccc} A = V/\Lambda & \xrightarrow{\phi_H} & V^*/\Lambda^* \\ \phi_X \downarrow & & \approx \downarrow \xi \to \chi_\xi \\ \mathrm{Pic}^0(A) & \xrightarrow{\approx} & \hat{\Lambda} \end{array}$$

It remains to exhibit a Riemann form on V^*/Λ^*. The map $V \to V^*$ (which we also denote by ϕ_H) given by $t \to H(t, \cdot)$ is an isomorphism as we have already pointed out. Define

$$H^*(\xi, \eta) = H(\phi_H^{-1}(\xi), \phi_H^{-1}(\eta)).$$

H^* is certainly a Hermitian form on V^* but $E^* = \mathrm{Im}\, H^*$ need not be integer valued on Λ^*. Using the above commutative diagram we see the kernel of $\phi_H\colon V/\Lambda \to V^*/\Lambda^*$ is finite implying $\phi_H^{-1}(\Lambda^*)/\Lambda$ is finite. Thus an appropriate integer multiple of H^* is a Riemann form on V^*/Λ^*. By "transport of structure," $\mathrm{Pic}^0(A)$ becomes an abelian manifold, $\hat{A}$ called the *dual abelian manifold* of A.

The association $A \leftrightarrow \hat{A}$ is a genuine duality. If $\rho\colon A \to B$ is a morphism of abelian manifolds, then $\hat{\rho}\colon \hat{B} \to \hat{A}$ is the morphism of abelian manifolds induced by pulling back divisors. One can show A is canonically isomorphic to $\hat{\hat{A}}$, etc.

For every ample divisor X on A, $\phi_X\colon A \to \hat{A}$ is an isogeny with kernel of order $\mathrm{Pf}(E)^2$ where E is the alternating Riemann form corresponding to X.

Before leaving the topic of dual abelian manifolds we briefly discuss the *Rosati involution*. If X is an ample divisor on A the isogeny $\phi_X \in \mathrm{Hom}(A, \hat{A})$ is an isomorphism in $\mathrm{Hom}_0(A, \hat{A})$. Let $\phi_X^{-1} \in \mathrm{Hom}_0(\hat{A}, A)$ be its inverse. If $\rho\colon A \to A$ let $\hat{\rho}\colon \hat{A} \to \hat{A}$ be the dual morphism. Then $\rho \to \hat{\rho}$ extends to a map from $\mathrm{End}_0(A) \to \mathrm{End}_0(\hat{A})$. For $\rho \in \mathrm{End}_0(A)$ define $\rho' = \phi_X^{-1} \circ \hat{\rho} \circ \phi_X$. The map $\rho \to \rho'$ is an involution, i.e. $(\rho_1 \circ \rho_2)' = \rho_2' \circ \rho_1'$ and $\rho'' = \rho$, called the Rosati involution on $\mathrm{End}_0(A)$ (we suppress the dependence on X in the notation). Let tr denote the trace map on the semisimple $\mathbb{Q}$-algebra $\mathrm{End}_0(A)$. Then $\mathrm{tr}(\rho'\rho) > 0$ for all $\rho \neq 0$. All this follows from

Proposition. *Let* X *be an ample divisor on* A, *and* H *the corresponding Riemann form. Then* $H(\rho' z, w) = H(z, \rho w)$ *for all* $z, w \in V$, *i.e.* ρ' *is the adjoint of* ρ *with respect to* H.

The proof, which is not hard, follows from carefully unwinding the definitions.

In the case of abelian varieties over finite fields, the existence and positivity of the Rosati involution can be used to prove the Riemann hypothesis for the associated zeta function (see [M–AV, Chap. IV, §21] for this). Another application is to classify the endomorphism rings of abelian manifolds. This theory was developed in the 1930s by A. A. Albert and others under the rubric of "Riemann matrices." For example, if $D = \mathrm{End}_0(A)$ is a division algebra, then the center of D is either totally real or else a CM field. [M–AV, Chap. IV, §21], gives the main results.

§5. Polarizations and Polarized Abelian Manifolds

For many purposes, it is natural to consider not just an abelian manifold A, but A together with the choice of a non-degenerate Riemann form. Roughly speaking, a polarized abelian manifold is a pair (A, H) where H is a non-degenerate Riemann form. We actually use a slightly different definition: say two Riemann forms H_1, H_2 are *equivalent* if there exists $n_1, n_2 \in \mathbb{N}$ such that $n_1 H_1 = n_2 H$. Then a *polarized abelian manifold* is an abelian manifold A together with an equivalence class of Riemann forms on A that contains a non-degenerate Riemann form. Such an equivalence class is called a (homogeneous) *polarization* of A. We use the notation $(A, \tilde{H})$ for a polarized abelian manifold, where $\tilde{H}$ is the equivalence class of the Riemann form H.

Note that a non-degenerate Riemann form on A corresponds to an algebraic equivalence class of a non-degenerate positive divisor. Such a divisor is ample on A, and gives rise to a projective embedding of A. Then a polarized abelian manifold $(A, \tilde{H})$ corresponds to giving the abelian manifold A together with an equivalence class of projective embeddings of A.

A morphism of polarized abelian manifolds $\phi\colon (A_1, \tilde{H}_1) \to (A_2, \tilde{H}_2)$ is a morphism $\phi\colon A_1 \to A_2$ such that $\phi^* H_2 \in \tilde{H}_1$.

A justification for the introduction of the notion of polarization is given by the following result.

Theorem. *The automorphism group of a polarized abelian manifold is finite.*

PROOF. Let $(A, \tilde{H})$ be a polarized abelian manifold, where H is a non-degenerate Riemann form. Suppose $\sigma \in \mathrm{Aut}(A, \tilde{H})$. Then $H(\sigma x, \sigma y) = H(x, y)$ (and not just an integer multiple of H). Here we have lifted σ to a linear map on V, the universal cover of A. Thus σ belongs to the compact group of linear maps preserving H. On the other hand, σ is determined by its restriction to the lattice Λ, on which it preserves the $\mathbb{Z}$-valued alternating form $E = \operatorname{Im} H$. So σ belongs to a discrete group also. But the intersection of a compact set with a discrete set is finite! □

A special role is played by the *principally-polarized abelian manifolds*. These are the polarized abelian manifolds $(A, \bar{H})$ for which there is $H \in \bar{H}$ with $\operatorname{Pf}(\operatorname{Im} H) = 1$. Thus with respect to a symplectic basis of Λ, $E = \operatorname{Im} H$ is given by the matrix

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Examples of principally polarized (p.p.) abelian manifolds are given by the *Jacobian varieties* of non-singular algebraic curves (or Riemann surfaces). If Γ is a non-singular algebraic curve of genus $g > 0$, defined over $\mathbb{C}$, then one can show that $\operatorname{Pic}^0(\Gamma) = \{\text{divisors of degree 0 on } \Gamma\}/\{\text{linear equivalence}\}$ has the structure of an abelian variety $J = \operatorname{Jac}(\Gamma)$, the Jacobian of Γ. (The Abel–Jacobi theorem, proved in [L–A, Chap. IV] shows that $\operatorname{Pic}^0(\Gamma)$ has the structure of a complex torus, while Riemann's relations, proved in [L–A, Chap. IV, §4], imply that $\operatorname{Pic}^0(\Gamma)$ is an abelian manifold). Fix $p_0 \in \Gamma$. Then the map $\alpha: \Gamma \to J$ given by $p \to [p - p_0]$ induces a map $\alpha^{(g-1)}: \Gamma^{(g-1)} \to J$ given by $\alpha^{(g-1)}(p_1, p_2, \ldots, p_{g-1}) = \sum_{j=1}^{g-1} \alpha(p_j)$. Here $\Gamma^{(g-1)}$ is the symmetric product of Γ with itself $g - 1$ times. The image of $\Gamma^{(g-1)}$ in J is well determined up to translation and is a non-degenerate divisor, the theta divisor θ. It can be shown that θ determines a principal polarization on J.

Torelli's theorem (see [G–H, p. 359] for a proof) says that the pair (J, θ) determines Γ up to isomorphism. More precisely, if $(J, \theta) \approx (J', \theta')$, where $J' = \operatorname{Jac}(\Gamma')$, then $\Gamma \approx \Gamma'$.

The following proposition is often useful.

Proposition. *Every polarized abelian manifold is isogenous to a principally polarized abelian manifold.*

PROOF. Let $(A, \bar{H})$ be a polarized abelian manifold of dimension d. As usual, $A = V/\Lambda$ and $E = \operatorname{Im} H$ is integer valued on Λ. Let $\{\lambda_1, \lambda_2, \ldots, \lambda_{2d}\}$ be a symplectic basis for Λ. In particular, $E(\lambda_j, \lambda_{d+j}) = e_j$, $1 \le j \le d$, for some integers e_j. Define a new lattice

$$\Lambda' = \sum_{j=1}^{d} \frac{1}{e_j} \lambda_j \mathbb{Z} + \sum_{j=1}^{d} \lambda_{d+j} \mathbb{Z}.$$

Then E as an alternating form on Λ', is integer valued, and has determinant 1. Let $A' = V/\Lambda'$. Then, $A = V/\Lambda \to V/\Lambda' = A'$ is an isogeny and A' is principally polarized (by E). □

If $(A, \bar{H})$ is a polarized abelian manifold, then $\phi_H: A \to \hat{A}$, the dual abelian manifold, is an isogeny with kernel of order $\det E = \operatorname{Pf}(E)^2$. In particular, if A is principally polarized, $A \approx \hat{A}$. It follows that Jacobians are self-dual.

§6. The Space of Principally Polarized Abelian Manifolds

For $d \ge 1$, let $\mathscr{A}_d$ be the set of isomorphism classes of principally polarized abelian manifolds. We will indicate how $\mathscr{A}_d$ can be given the structure of a complex analytic space.

Consider the case $d = 1$. Every abelian manifold of dimension 1 is principally polarized, so the polarization is irrelevant. Let $\mathscr{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im} \tau > 0\}$, the Poincaré upper half plane. One has $\mathscr{A}_1 \approx \operatorname{Sl}_2(\mathbb{Z}) \backslash \mathscr{H}$ given by

$$\mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \to \tau = \omega_1/\omega_2$$

where $\operatorname{Im}(\omega_1/\omega_2) > 0$. The action of $\operatorname{Sl}_2(\mathbb{Z})$ on $\mathscr{H}$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(\tau) = \frac{a\tau + b}{c\tau + d}.$$

This material, which is fairly familiar, will be seen as a special case of our further considerations.

If $A = V/\Lambda$ is a principally polarized abelian variety we choose a $\mathbb{C}$-basis for V and a symplectic basis for Λ. One then sees that A has a concrete representation as $(\mathbb{C}^d/\langle \omega_1, \ldots, \omega_{2d} \rangle, H)$ where $\omega_1, \ldots, \omega_{2d}$ are $\mathbb{R}$-linearly independent column vectors and H is a Riemann form whose imaginary part E has matrix $(E(\omega_i, \omega_j)) = J$ where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. The $d \times 2d$ complex matrix $\Omega = (\omega_1, \omega_2, \ldots, \omega_{2d})$ is called the *period matrix*.

Two questions arise.

(1) What conditions on Ω express the condition that the alternating form E on $\langle \omega_1, \omega_2, \ldots, \omega_{2d} \rangle$ given by J is the imaginary part of a non-degenerate Riemann form?

(2) When do two period matrices Ω and Ω' correspond to isomorphic principally polarized abelian manifolds?

Some calculations (see [L–A, Chap. VIII, §1]) show that Ω determines an abelian manifold if and only if the following two conditions hold.

(RI) $\qquad \Omega J \Omega^t = 0 \qquad (\Omega^t = \text{transpose of } \Omega),$

(RII) $\qquad 2i(\bar{\Omega} J^{-1} \Omega^t)^{-1} > 0 \qquad (>0 \text{ means positive definite}).$

These conditions are known as Riemann's relations. (RI) is equivalent to the condition $E(iz, iw) = E(z, w)$, and (RII) is equivalent to the condition that H be positive definite. In fact, the matrix of H with respect to the standard basis of $\mathbb{C}^d$ is $2i(\bar{\Omega} J^{-1} \Omega^t)^{-1}$.

If we write $\Omega = (\Omega_1, \Omega_2)$ with Ω_1, Ω_2 complex $d \times d$ matrices, the Riemann relations take the form

$$\text{(RI}') \qquad \Omega_2\Omega_1^t - \Omega_1\Omega_2^t = 0,$$

$$\text{(RII}') \qquad 2i(\Omega_2\bar{\Omega}_1^t - \Omega_1\bar{\Omega}_2^t) > 0.$$

From these relations it is easy to show that both Ω_1 and Ω_2 are invertible.

Let $\mathscr{R}$ be the set of complex matrices $\Omega = (\Omega_1 \Omega_2)$ satisfying (RI′) and (RII′). $\mathrm{Gl}_d(\mathbb{C})$ acts on $\mathscr{R}$ by multiplication on the left and $\mathrm{Sp}_{2d}(\mathbb{Z}) = \{M \in \mathrm{Gl}_{2d}(\mathbb{Z}) | MJM^t = J\}$ acts on $\mathscr{R}$ by multiplication on the right. One sees that

$$\mathscr{A}_d \approx \mathrm{Gl}_d(\mathbb{C})\backslash\mathscr{R}/\mathrm{Sp}_{2d}(\mathbb{Z}).$$

The action of $\mathrm{Gl}_d(\mathbb{C})$ gives an isomorphism of abelian varieties, while the action of $\mathrm{Sp}_{2d}(\mathbb{Z})$ corresponds to a change of symplectic basis.

Thus $\Omega = (\Omega_1 \Omega_2) \sim (\tau, I)$ where $\tau = \Omega_2^{-1}\Omega_1$. The conditions (RI′) and (RII′) assert that τ is symmetric and $\mathrm{Im}\,\tau$ is positive definite.

Definition. Let $\mathscr{H}_d$ be the space of $d \times d$ complex matrices τ which are symmetric and $\mathrm{Im}\,\tau$ is positive definite. $\mathscr{H}_d$ is called *the Siegel upper half space*. It is a complex manifold of dimension $d(d + 1)/2$.

When is $(\tau, I) \sim (\tau', I)$? Write $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Then, $(\tau, I)M = (\tau A + C, \tau B + D) \sim ((\tau B + D)^{-1}(\tau A + C), I)$. Thus, we must have

$$\tau' = (\tau B + D)^{-1}(\tau A + C)$$

for some $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in $\mathrm{Sp}_{2d}(\mathbb{Z})$.

To put things in somewhat more familiar form we note that $\mathrm{Sp}_{2d}(\mathbb{Z})$ is invariant under transpose and that $\tau^t = \tau$. Thus, we let $\mathrm{Sp}_{2d}(\mathbb{Z})$ act on $\mathscr{H}_d$ by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \tau \to (A\tau + B)(C\tau + D)^{-1}.$$

The final conclusion is

$$\boxed{\mathscr{A}_d \approx \mathrm{Sp}_{2d}(\mathbb{Z})\backslash\mathscr{H}_d.}$$

By analyzing this more carefully we see $\mathscr{A}_d$ is parametrized by a $d(d + 1)/2$ complex analytic space. In fact, this space can be given the structure of a normal quasi-projective variety.

When $d = 1$ we recover the familiar $\mathscr{A}_1 \approx \mathrm{Sl}_2(\mathbb{Z})\backslash\mathscr{H}$ which, via the j-function, is isomorphic to the complex plane.

If one investigates in a similar manner the "space" of complex tori of

dimension d it turns out to depend on d^2 complex parameters. Thus one suspects that when $d > 1$ there are complex tori which are not abelian varieties since

$$d^2 - \frac{d(d + 1)}{2} = \frac{d(d - 1)}{2} \geq 1 \quad \text{when } d \geq 2.$$

This is indeed the case.

As an example, let $d = 2$ and set

$$\Omega = \begin{pmatrix} \alpha + i & \beta & 1 & 0 \\ \gamma & \delta + i & 0 & 1 \end{pmatrix},$$

where α, β, γ, and δ are real and algebraically independent over $\mathbb{Q}$. Let T be the torus $\mathbb{C}^2/\langle\Omega\rangle$, where $\langle\Omega\rangle$ is the lattice generated by the columns of Ω. We will show $\mathscr{M}(T) = \mathbb{C}$. Suppose $f \in \mathscr{M}(T)$ is not a constant. Then $(f) = D_0 - D_\infty$ where D_∞ is a non-zero positive divisor on T. As shown in [SHAF, pp. 354–356], D_∞ corresponds to an integral, non-zero, antisymmetric matrix A such that $\Omega A \Omega^t = 0$. Writing this out shows there is a non-trivial linear relation with coefficients in $\mathbb{Z}$ between $1, \beta, \gamma, \alpha + i, \delta + i$, and $\alpha\delta - \beta\gamma + i(\alpha + \delta)$ and so between $1, \beta, \gamma, \alpha, \delta, \alpha\delta - \beta\gamma$. This contradicts the algebraic independence of α, β, γ, and δ.

Here is a heuristic argument that the generic abelian variety is simple. If A is a principally polarized abelian variety of dimension d which is not simple then A is isogenous to $B \times C$ where $\dim B = b$ and $\dim C = c$ with $1 \leq b$, $c < d$, and $b + c = d$. The modulus of B is in $\mathscr{A}_b$, a $[b(b + 1)/2]$-dimensional space, and the modulus of C is in $\mathscr{A}_c$, a $[c(c + 1)/2]$-dimensional space. Thus the moduli of A with a factor of dimension b lie in a subspace of $\mathscr{A}_d$ of dimension

$$\frac{b(b + 1)}{2} + \frac{(d - b)(d - b + 1)}{2}.$$

The maximum of these dimensions for $1 \leq b < d$ is easily seen to be that given by $b = [d/2]$, namely $[(d^2 + 2d + 1)/4]$ (here $[x]$ means the greatest integer $\leq x$). But,

$$\left[\frac{d^2 + 2d + 1}{4}\right] < \frac{d(d + 1)}{2} \quad \text{for all } d > 1.$$

Finally, we show the generic abelian variety A has $\mathbb{Z}$ for its endomorphism ring. Let $\tau \in \mathscr{A}_d$ and A be the abelian variety corresponding to (τ, I). Then multiplication by $g \in M_d(\mathbb{C})$ gives an element of $\mathrm{End}(A)$ if and only if there is a matrix $M \in M_{2d}(\mathbb{Z})$ such that $g(\tau, I) = (\tau, I)M$. Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. We must have $(g\tau, g) = (\tau A + C, \tau B + D)$ so $g = \tau B + D$ and $g\tau = \tau A + C$. Thus, $\tau B\tau + D\tau - \tau A - C = 0$. Let $\tau = (t_{ij})$. If $B = C = 0$ and $A = D = nI$ for $n \in \mathbb{Z}$ there is no condition imposed on τ. Otherwise the t_{ij} must satisfy certain non-trivial quadratic polynomials with coefficients in $\mathbb{Z}$. In general, this cannot happen. For example, suppose t_{ij} with $i \leq j$ are algebraically independent over $\mathbb{Q}$. Then a simple calculation shows that $\tau B\tau + D\tau - \tau A - C = 0$ can only happen when $C = B = 0$ and $A = D = nI$ for some $n \in \mathbb{Z}$. For the corresponding abelian variety the endomorphism ring is precisely $\mathbb{Z}$. Of course, when $d = 1$ we need only require that $t = t_{11}$ not be quadratic over $\mathbb{Q}$.

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