1
L-functions

1.1 L-functions Attached to Modular Forms

Let $f = \sum_{n\geq 1} a_n q^n \in S_k(\Gamma_1(N))$ be a cusp form.

Definition 1.1.1 (L-series). The *L-series* of f is

$$L(f,s) = \sum_{n\geq 1} \frac{a_n}{n^s}.$$

Definition 1.1.2 (Λ-function). The *completed* Λ *function* of f is

$$\Lambda(f,s) = N^{s/2}(2\pi)^{-s}\Gamma(s)L(f,s),$$

where

$$\Gamma(s) = \int_0^\infty e^{-t}t^s \frac{dt}{t}$$

is the Γ function (so $\Gamma(n) = (n-1)!$ for positive integers n).

We can view $\Lambda(f,s)$ as a (Mellin) transform of f, in the following sense:

Proposition 1.1.3. *We have*

$$\Lambda(f,s) = N^{s/2}\int_0^\infty f(iy)y^s \frac{dy}{y},$$

and this integral converges for $\mathrm{Re}(s) > \frac{k}{2}+1$.

Proof. We have

$$\begin{aligned}\int_0^\infty f(iy)y^s\frac{dy}{y} &= \int_0^\infty \sum_{n=1}^\infty a_n e^{-2\pi n y} y^s \frac{dy}{y}\\ &= \sum_{n=1}^\infty a_n \int_0^\infty e^{-t}(2\pi n)^{-s}t^s\frac{dt}{t} \qquad (t = 2\pi n y)\\ &= (2\pi)^{-s}\Gamma(s)\sum_{n=1}^\infty \frac{a_n}{n^s}.\end{aligned}$$

To go from the first line to the second line, we reverse the summation and integration and perform the change of variables $t = 2\pi n y$. (We omit discussion of convergence.) □

1.1.1 Analytic Continuation and Functional Equation

We define the *Atkin-Lehner operator* W_N on $S_k(\Gamma_1(N))$ as follows. If $w_N = \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$, then $[w_N^2]_k$ acts as $(-N)^{k-2}$, so if

$$W_N(f) = N^{1-\frac{k}{2}} \cdot f|[w_N]_k,$$

then $W_N^2 = (-1)^k$. (Note that W_N is an involution when k is even.) It is easy to check directly that if $\gamma \in \Gamma_1(N)$, then $w_N \gamma w_N^{-1} \in \Gamma_1(N)$, so W_N preserves $S_k(\Gamma_1(N))$. Note that in general W_N does *not* commute with the Hecke operators T_p, for $p \mid N$.

The following theorem is mainly due to Hecke (and maybe other people, at least in this generality). For a very general version of this theorem, see [Li75].

Theorem 1.1.4. *Suppose $f \in S_k(\Gamma_1(N), \chi)$ is a cusp form with character χ. Then $\Lambda(f, s)$ extends to an entire (holomorphic on all of $\mathbf{C}$) function which satisfies the functional equation*

$$\Lambda(f, s) = i^k \Lambda(W_N(f), k - s).$$

Since $N^{s/2}(2\pi)^{-s}\Gamma(s)$ is everywhere nonzero, Theorem 1.1.4 implies that $L(f, s)$ also extends to an entire function.

It follows from Definition 1.1.2 that $\Lambda(cf, s) = c\Lambda(f, s)$ for any $c \in \mathbf{C}$. Thus if f is a W_N-eigenform, so that $W_N(f) = wf$ for some $w \in \mathbf{C}$, then the functional equation becomes

$$\Lambda(f, s) = i^k w \Lambda(f, k - s).$$

If $k = 2$, then W_N is an involution, so $w = \pm 1$, and the sign in the functional equation is $\varepsilon(f) = i^k w = -w$, which is the negative of the sign of the Atkin-Lehner involution W_N on f. It is straightforward to show that $\varepsilon(f) = 1$ if and only if $\mathrm{ord}_{s=1} L(f, s)$ is even. Parity observations such as this are extremely useful when trying to understand the Birch and Swinnerton-Dyer conjecture.

Sketch of proof of Theorem 1.1.4 when $N = 1$. We follow [Kna92, §VIII.5] closely.

Note that since $w_1 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$, the condition $W_1(f) = f$ is satisfied for any $f \in S_k(1)$. This translates into the equality

$$f\left(-\frac{1}{z}\right) = z^k f(z). \qquad (1.1.1)$$

Write $z = x + iy$ with x and y real. Then (1.1.1) along the positive imaginary axis (so $z = iy$ with y positive real) is

$$f\left(\frac{i}{y}\right) = i^k y^k f(iy). \tag{1.1.2}$$

From Proposition 1.1.3 we have

$$\Lambda(f,s) = \int_0^\infty f(iy) y^{s-1} dy, \tag{1.1.3}$$

and this integral converges for $\mathrm{Re}(s) > \frac{k}{2} + 1$.

Again using growth estimates, one shows that

$$\int_1^\infty f(iy) y^{s-1} dy$$

converges for all $s \in \mathbf{C}$, and defines an entire function. Breaking the path in (1.1.3) at 1, we have for $\mathrm{Re}(s) > \frac{k}{2} + 1$ that

$$\Lambda(f,s) = \int_0^1 f(iy) y^{s-1} dy + \int_1^\infty f(iy) y^{s-1} dy.$$

Apply the change of variables $t = 1/y$ to the first term and use (1.1.2) to get

$$\begin{aligned}\int_0^1 f(iy) y^{s-1} dy &= \int_\infty^1 -f(i/t) t^{1-s} \frac{1}{t^2} dt \\ &= \int_1^\infty f(i/t) t^{-1-s} dt \\ &= \int_1^\infty i^k t^k f(it) t^{-1-s} dt \\ &= i^k \int_1^\infty f(it) t^{k-1-s} dt.\end{aligned}$$

Thus

$$\Lambda(f,s) = i^k \int_1^\infty f(it) t^{k-s-1} dt + \int_1^\infty f(iy) y^{s-1} dy.$$

The first term is just a translation of the second, so the first term extends to an entire function as well. Thus $\Lambda(f,s)$ extends to an entire function.

The proof of the general case for $\Gamma_0(N)$ is almost the same, except the path is broken at $1/\sqrt{N}$, since $i/\sqrt{N}$ is a fixed point for w_N. □

1.1.2 A Conjecture About Nonvanishing of $L(f, k/2)$

Suppose $f \in S_k(1)$ is an eigenform. If $k \equiv 2 \pmod 4$, then $L(f, k/2) = 0$ for reasons related to the discussion after the statement of Theorem 1.1.4. On the other hand, if $k \equiv 0 \pmod 4$, then $\mathrm{ord}_{s=k/2} L(f, k/2)$ is even, so $L(f, k/2)$ may or may not vanish.

Conjecture 1.1.5. *Suppose $k \equiv 0 \pmod 4$. Then $L(f, k/2) \neq 0$.*

According to [CF99], Conjecture 1.1.5 is true for weight k if there is some n such that the characteristic polynomial of T_n on $S_k(1)$ is irreducible. Thus Maeda's conjecture implies Conjecture 1.1.5. Put another way, if you find an f of level 1 and weight $k \equiv 0 \pmod 4$ such that $L(f, k/2) = 0$, then Maeda's conjecture is false for weight k.

Oddly enough, I personally find Conjecture 1.1.5 less convincing that Maeda's conjecture, despite it being a weaker conjecture.

1.1.3 Euler Products

Euler products make very clear how L-functions of eigenforms encode deep arithmetic information about representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. Given a "compatible family" of ℓ-adic representations ρ of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, one can define an Euler product $L(\rho, s)$, but in general it is very hard to say anything about the analytic properties of $L(\rho, s)$. However, as we saw above, when ρ is attached to a modular form, we know that $L(\rho, s)$ is entire.

Theorem 1.1.6. *Let $f = \sum a_n q^n$ be a newform in $S_k(\Gamma_1(N), \varepsilon)$, and let $L(f, s) = \sum_{n\geq 1} a_n n^{-s}$ be the associated Dirichlet series. Then $L(f, s)$ has an Euler product*

$$L(f, s) = \prod_{p\mid N} \frac{1}{1 - a_p p^{-s}} \cdot \prod_{p\nmid N} \frac{1}{1 - a_p p^{-s} + \varepsilon(p) p^{k-1} p^{-2s}}.$$

Note that it is not really necessary to separate out the factors with $p \mid N$ as we have done, since $\varepsilon(p) = 0$ whenever $p \mid N$. Also, note that the denominators are of the form $F(p^{-s})$, where

$$F(X) = 1 - a_p X + \varepsilon(p) p^{k-1} X^2$$

is the reverse of the characteristic polynomial of Frob_p acting on any of the ℓ-adic representations attached to f, with $p \neq \ell$.

Recall that if p is a prime, then for every $r \geq 2$ the Hecke operators satisfy the relationship

$$T_{p^r} = T_{p^{r-1}} T_p - p^{k-1} \varepsilon(p) T_{p^{r-2}}. \tag{1.1.4}$$

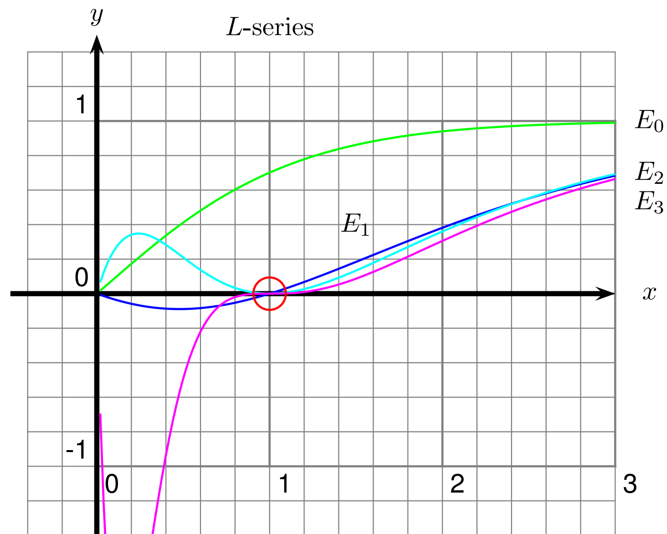
Lemma 1.1.7. *For every prime p we have the formal equality*

$$\sum_{r\geq 0} T_{p^r} X^r = \frac{1}{1 - T_p X + \varepsilon(p) p^{k-1} X^2}. \tag{1.1.5}$$

Proof. Multiply both sides of (1.1.5) by $1 - T_p X + \varepsilon(p) p^{k-1} X^2$ to obtain the equation

$$\sum_{r\geq 0} T_{p^r} X^r - \sum_{r\geq 0} (T_{p^r} T_p) X^{r+1} + \sum_{r\geq 0} (\varepsilon(p) p^{k-1} T_{p^r}) X^{r+2} = 1.$$

This equation is true if and only if the lemma is true. Equality follows by checking the first few terms and shifting the index down by 1 for the second sum and down by 2 for the third sum, then using (1.1.4). □



$E_0 = [0,0,0,0,1]$, $E_1 = [0,0,1,-1,0]$, $E_2 = [0,1,1,-2,0]$, $E_3 = [0,0,1,-7,6]$

FIGURE 1.1.1. Graph of $L(E,s)$ for s real, for curves of ranks 0 to 3.

Note that $\varepsilon(p) = 0$ when $p \mid N$, so when $p \mid N$

$$\sum_{r \geq 0} T_{p^r} X^r = \frac{1}{1 - T_p X}.$$

Since the eigenvalues a_n of f also satisfy (1.1.4), we obtain each factor of the Euler product of Theorem 1.1.6 by substituting the a_n for the T_n and p^{-s} for X into (1.1.4). For $(n,m) = 1$, we have $a_{nm} = a_n a_m$, so

$$\sum_{n \geq 1} \frac{a_n}{n^s} = \prod_p \left(\sum_{r \geq 0} \frac{a_{p^r}}{p^{rs}} \right),$$

which gives the full Euler product for $L(f,s) = \sum a_n n^{-s}$.

1.1.4 Visualizing L-function

A. Shwayder did his Harvard junior project with me on visualizing L-functions of elliptic curves (or equivalently, of newforms $f = \sum a_n q^n \in S_2(\Gamma_0(N))$ with $a_n \in \mathbf{Z}$ for all n. The graphs in Figures 1.1.1–1.1.2 of $L(E,s)$, for s real, and $|L(E,s)|$, for s complex, are from his paper.

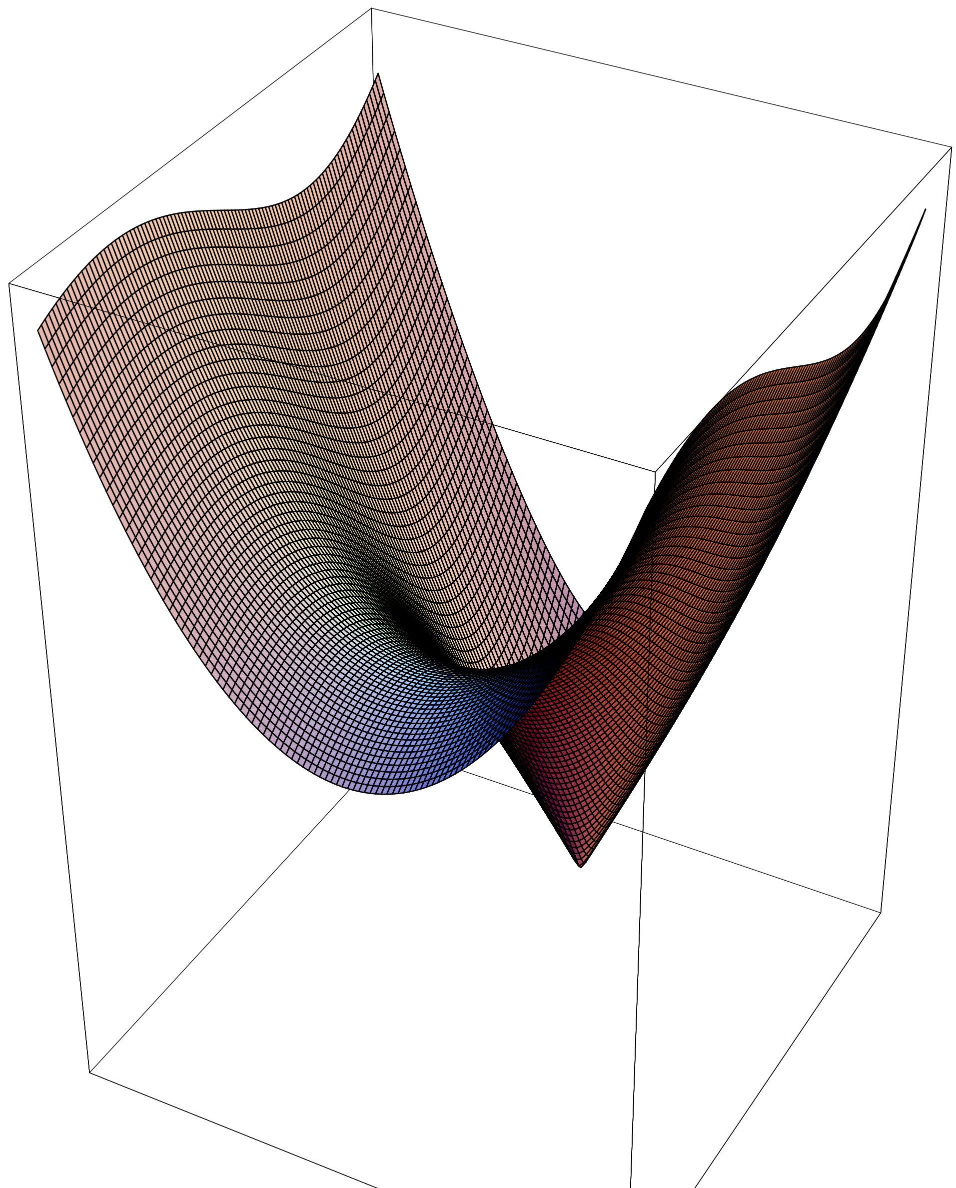


FIGURE 1.1.2. Graph of $|L(E,s)|$, for s complex for $y^2 + y = x^3 - x^2 - 10x - 20$

References

[CF99] J. B. Conrey and D. W. Farmer, *Hecke operators and the nonvanishing of L-functions*, Topics in number theory (University Park, PA, 1997), Math. Appl., vol. 467, Kluwer Acad. Publ., Dordrecht, 1999, pp. 143–150. MR 2000f:11055

[Kna92] A. W. Knapp, *Elliptic curves*, Princeton University Press, Princeton, NJ, 1992.

[Li75] W-C. Li, *Newforms and functional equations*, Math. Ann. **212** (1975), 285–315.