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Modular Forms of Level 1 (W. Stein, Math 252, 10/27/03)

In this chapter, we view modular forms of level 1 both as holomorphic functions on the upper half plane and functions on lattices. We then define Hecke operators on modular forms, and derive explicit formulas for the action of Hecke operators on q-expansions. An excellent reference for the theory of modular forms of level 1 is Serre [9, Ch. 7].

1.1 The Definition

Let k be an integer. The space $S_k = S_k(1)$ of cusp forms of level 1 and weight k consists of all functions f that are holomorphic on the upper half plane $\mathfrak{h}$ and such that for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$ one has

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau), \tag{1.1.1}$$

and f vanishes at infinity, in a sense which we will now make precise. The matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ is in $\mathrm{SL}_2(\mathbf{Z})$, so $f(\tau+1) = f(\tau)$. Thus f passes to a well-defined function of $q(\tau) = e^{2\pi i\tau}$. Since for $\tau \in \mathfrak{h}$ we have $|q(\tau)| < 1$, we may view $f = f(q)$ as a function of q on the punctured open unit disc $\{q : 0 < |q| < 1\}$. The condition that $f(\tau)$ vanishes at infinity means that $f(q)$ extends to a holomorphic function on the open disc $\{z : |z| < 1\}$ so that $f(0) = 0$. Because holomorphic functions are represented by power series, there is a neighborhood of 0 such that

$$f(q) = \sum_{n=1}^{\infty} a_n q^n,$$

so for all $\tau \in \mathfrak{h}$ with sufficiently large imaginary part, $f(\tau) = \sum_{n=1}^{\infty} a_n e^{2\pi i n\tau}$.

It will also be useful to consider the slightly large space $M_k(1)$ of holomorphic functions on $\mathfrak{h}$ that transform as above and are merely required to be holomorphic at infinity.

Remark 1.1.1. In fact, the series $\sum_{n=1}^{\infty} a_n e^{2\pi i n\tau}$ converges for all $\tau \in \mathfrak{h}$. This is because the Fourier coefficients a_n are $O(n^{k/2})$ (see [8, Cor. 2.1.6, pg. 43]).

Remark 1.1.2. In [9, Ch. 7], the weight is defined in the same way, but in the notation our k is twice his k.

1.2 Some Examples and Conjectures

The space $S_k(1)$ of cusp forms is a finite-dimensional complex vector space. For k even we have $\dim S_k(1) = \lfloor k/12 \rfloor$ if $k \not\equiv 2 \pmod{12}$ and $\lfloor k/12 \rfloor - 1$ if $k \equiv 2 \pmod{12}$, except when $k = 2$ in which case the dimension is 0. For even k, the space $M_k(1)$ has dimension 1 more than the dimension of $S_k(1)$, except when $k = 2$ when both have dimension 0. (For proofs, see, e.g., [9, Ch. 7, §3].)

By the dimension formula mentioned above, the first interesting example is the space $S_{12}(1)$, which is a 1-dimensional space spanned by

$$\begin{aligned}\Delta(q) &= q \prod_{n=1}^{\infty} (1-q^n)^{24} \\ &= q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 - 16744q^7 + 84480q^8 \\ &\quad - 113643q^9 - 115920q^{10} + 534612q^{11} - 370944q^{12} - 577738q^{13} + \cdots\end{aligned}$$

That Δ lies in $S_{12}(1)$ is proved in [9, Ch. 7, §4.4] by expressing Δ in terms of elements of $M_4(1)$ and $M_6(1)$, and computing the q-expansion of the resulting expression.

The Ramanujan τ function $\tau(n)$ assigns to n the nth coefficient of $\Delta(q)$.

Conjecture 1.2.1 (Lehmer). *$\tau(n) \neq 0$ for all $n \geq 1$.*

This conjecture has been verified for $n \leq 22689242781695999$ (see Jordan and Kelly, 1999).

Conjecture 1.2.2 (Edixhoven). *Let p be a prime. There a polynomial time algorithm to compute $\tau(p)$, polynomial in the number of digits of p.*

Edixhoven has proposed an approach to find such an algorithm. His idea is to use ℓ-adic cohomology to find an analogue of the Schoof-Elkies-Atkin algorithm (which counts the number N_q of points on an elliptic curves over a finite field $\mathbf{F}_q$ by computing $N_q \mod \ell$ for many primes ℓ). Here's what Edixhoven has to say about the status of his conjecture (email, October 22, 2003):

> I have made a lot of progress on proving that my method runs in polynomial time, but it is not yet complete. I expect that all should be completed in 2004. For higher weights [...] you need to compute on varying curves such as $X_1(\ell)$ for ℓ up to $\log(p)$ say.
>
> An important by-product of my method is the computation of the mod ℓ Galois representations associated to Δ in time polynomial in ℓ. So, it should be seen as an attempt to make the Langlands correspondence for GL_2 over $\mathbf{Q}$ available computationally.

If $f \in M_k(1)$ and $g \in M_{k'}(1)$, then it is easy to see from the definitions that $fg \in M_{k+k'}(1)$. Moreover, $\oplus_{k\geq 0} M_k(1)$ is a commutative graded ring generated freely by $E_4 = 1 + 240\sum_{n=1}^{\infty} \sigma_3(n)q^n$ and $E_6 = 1 - 504\sum_{n=1}^{\infty} \sigma_5(n)q^n$, where $\sigma_d(n)$ is the sum of the dth powers of the positive divisors of n (see [9, Ch.7, §3.2]).

Example 1.2.3. Because E_4 and E_6 generate, it is straightforward to write down a basis for any space $M_k(1)$. For example, the space $M_{36}(1)$ has basis

$$\begin{aligned}
f_1 &= 1 + 6218175600q^4 + 15281788354560q^5 + \cdots \\
f_2 &= q + 57093088q^4 + 37927345230q^5 + \cdots \\
f_3 &= q^2 + 194184q^4 + 7442432q^5 + \cdots \\
f_4 &= q^3 - 72q^4 + 2484q^5 + \cdots
\end{aligned}$$

1.3 Modular Forms as Functions on Lattices

In order to define Hecke operators, it will be useful to view modular forms as functions on lattices in $\mathbf{C}$.

A *lattice* $L \subset \mathbf{C}$ is a subring $L = \mathbf{Z}\omega_1 + \mathbf{Z}\omega_2$ for which $\omega_1, \omega_2 \in \mathbf{C}$ are linearly independent over $\mathbf{R}$. We may assume that $\omega_1/\omega_2 \in \mathfrak{h} = \{z \in \mathbf{C} : \mathrm{Im}(z) > 0\}$. Let $\mathcal{R}$ be the set of all lattices in $\mathbf{C}$. Let $\mathcal{E}$ be the set of isomorphism classes of pairs (E, ω), where E is an elliptic curve over $\mathbf{C}$ and $\omega \in \Omega^1_E$ is a nonzero holomorphic differential 1-form on E. Two pairs (E, ω) and (E', ω') are isomorphic if there is an isomorphism $\varphi : E \to E'$ such that $\varphi^*(\omega') = \omega$.

Proposition 1.3.1. *There is a bijection between $\mathcal{R}$ and $\mathcal{E}$ under which $L \in \mathcal{R}$ corresponds to $(\mathbf{C}/L,\, dz) \in \mathcal{E}$.*

Proof. We describe the maps in each direction, but leave the proof that they induce a well-defined bijection as an exercise for the reader. Given $L \in \mathcal{R}$, by Weierstrass theory the quotient $\mathbf{C}/L$ is an elliptic curve, which is equipped with the distinguished differential ω induced by the differential dz on $\mathbf{C}$.

Conversely, if E is an elliptic curve over $\mathbf{C}$ and $\omega \in \Omega^1_E$ is a nonzero differential, we obtain a lattice L in $\mathbf{C}$ by integrating homology classes:

$$L = L_\omega = \left\{ \int_\gamma \omega : \gamma \in \mathrm{H}_1(E(\mathbf{C}), \mathbf{Z}) \right\}.$$

□

Let

$$\mathcal{B} = \{(\omega_1, \omega_2) : \omega_1, \omega_2 \in \mathbf{C},\, \omega_1/\omega_2 \in \mathfrak{h}\},$$

be the set of ordered basis of lattices in $\mathbf{C}$, ordered so that $\omega_1/\omega_2 \in \mathfrak{h}$. There is a left action of $\mathrm{SL}_2(\mathbf{Z})$ on $\mathcal{B}$ given by

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(\omega_1, \omega_2) \mapsto (a\omega_1 + b\omega_2, c\omega_1 + d\omega_2)$$

and $\mathrm{SL}_2(\mathbf{Z})\backslash\mathcal{B} \cong \mathcal{R}$. (The action is just the left action of matrices on column vectors, except we write (ω_1, ω_2) as a row vector since it takes less space.)

Give a modular form $f \in M_k(1)$, associate to f a function $F : \mathcal{R} \to \mathbf{C}$ as follows. First, on lattices of the special form $\mathbf{Z}\tau + \mathbf{Z}$, for $\tau \in \mathfrak{h}$, let $F(\mathbf{Z}\tau + \mathbf{Z}) = f(\tau)$.

In order to extend F to a function on all lattices, suppose further that F satisfies the homogeneity condition $F(\lambda L) = \lambda^{-k} F(L)$, for any $\lambda \in \mathbf{C}$ and $L \in \mathcal{R}$. Then

$$F(\mathbf{Z}\omega_1 + \mathbf{Z}\omega_2) = \omega_2^{-k} F(\mathbf{Z}\omega_1/\omega_2 + \mathbf{Z}) := \omega_2^{-k} f(\omega_1/\omega_2).$$

That F is well-defined exactly amounts to the transformation condition (1.1.1) that f satisfies.

Lemma 1.3.2. *The lattice function $F : \mathcal{R} \to \mathbf{C}$ associated to $f \in M_k(1)$ is well defined.*

Proof. Suppose $\mathbf{Z}\omega_1 + \mathbf{Z}\omega_2 = \mathbf{Z}\omega_1' + \mathbf{Z}\omega_2'$ with ω_1/ω_2 and ω_1'/ω_2' both in $\mathfrak{h}$. We must verify that $\omega_2^{-k} f(\omega_1/\omega_2) = (\omega_2')^{-k} f(\omega_1'/\omega_2')$. There exists $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$ such that $\omega_1' = a\omega_1 + b\omega_2$ and $\omega_2' = c\omega_1 + d\omega_2$. Dividing, we see that $\omega_1'/\omega_2' = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(\omega_1/\omega_2)$. Because f is a weight k modular form, we have

$$f\left(\frac{\omega_1'}{\omega_2'}\right) = f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\left(\frac{\omega_1}{\omega_2}\right)\right) = \left(c\frac{\omega_1}{\omega_2} + d\right)^k f\left(\frac{\omega_1}{\omega_2}\right).$$

Multiplying both sides by ω_2^k yields

$$\omega_2^k f\left(\frac{\omega_1'}{\omega_2'}\right) = (c\omega_1 + d\omega_2)^k f\left(\frac{\omega_1}{\omega_2}\right).$$

Observing that $\omega_2' = c\omega_1 + d\omega_2$ and dividing again completes the proof. $\square$

Since $f(\tau) = F(\mathbf{Z}\tau + \mathbf{Z})$, we can recover f from F, so the map $f \mapsto F$ is injective. Moreover, it is surjective in the sense that if F is homogeneous of degree $-k$, then F arises from a function $f : \mathfrak{h} \to \mathbf{C}$ that transforms like a modular form. More precisely, if $F : \mathcal{R} \to \mathbf{C}$ satisfies the homogeneity condition $F(\lambda L) = \lambda^{-k} F(L)$, then the function $f : \mathfrak{h} \to \mathbf{C}$ defined by $f(\tau) = F(\mathbf{Z}\tau + \mathbf{Z})$ transforms like a modular form of weight k, as the following computation shows: For any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$ and $\tau \in \mathfrak{h}$, we have

$$\begin{aligned} f\left(\frac{a\tau + b}{c\tau + d}\right) &= F\left(\mathbf{Z}\frac{a\tau + b}{c\tau + d} + \mathbf{Z}\right) \\ &= F((c\tau + d)^{-1}(\mathbf{Z}(a\tau + b) + \mathbf{Z}(c\tau + d))) \\ &= (c\tau + d)^k F(\mathbf{Z}(a\tau + b) + \mathbf{Z}(c\tau + d)) \\ &= (c\tau + d)^k F(\mathbf{Z}\tau + \mathbf{Z}) \\ &= (c\tau + d)^k f(\tau). \end{aligned}$$

Say that a function $F : \mathcal{R} \to \mathbf{C}$ is holomorphic on $\mathfrak{h} \cup \infty$ if the function $f(\tau) = F(\mathbf{Z}\tau + \mathbf{Z})$ is. We summarize the above discussion in a proposition.

Proposition 1.3.3. *There is a bijection between $M_k(1)$ and functions $F : \mathcal{R} \to \mathbf{C}$ that are homogeneous of degree $-k$ and holomorphic on $\mathfrak{h} \cup \{\infty\}$. Under this bijection $F : \mathcal{R} \to \mathbf{C}$ corresponds to $f(\tau) = F(\mathbf{Z}\tau + \mathbf{Z})$.*

1.4 Hecke Operators

Define a map T_n from the free abelian group generated by all $\mathbf{C}$-lattices into itself by

$$T_n(L) = \sum_{\substack{L' \subset L \\ [L:L'] = n}} L',$$

where the sum is over all sublattices $L' \subset L$ of index n. For any function $F : \mathcal{R} \to \mathbf{C}$ on lattices, define $T_n(F) : \mathcal{R} \to \mathbf{C}$ by

$$(T_n(F))(L) = n^{k-1} \sum_{\substack{L' \subset L \\ [L:L'] = n}} F(L').$$

Note that if F is homogeneous of degree $-k$, then $T_n(F)$ is also homogeneous of degree $-k$.

Since $(n,m)=1$ implies $T_nT_m = T_{nm}$ and T_{p^k} is a polynomial in $\mathbf{Z}[T_p]$ (see [9, Cor. 1, pg. 99]), the essential case to consider is n prime.

Suppose $L' \subset L$ with $[L : L'] = n$. Then every element of L/L' has order dividing n, so $nL \subset L' \subset L$ and

$$L'/nL \subset L/nL \approx (\mathbf{Z}/n\mathbf{Z})^2.$$

Thus the subgroups of $(\mathbf{Z}/n\mathbf{Z})^2$ of order n correspond to the sublattices L' of L of index n. When $n = \ell$ is prime, there are $\ell+1$ such subgroups, since the subgroups correspond to nonzero vectors in $\mathbf{F}_\ell$ modulo scalar equivalence, and there are $(\ell^2-1)(\ell-1) = \ell+1$ of them.

Recall from Proposition 1.3.1 that there is a bijection between the set $\mathcal{R}$ of lattices in $\mathbf{C}$ and the set $\mathcal{E}$ of isomorphism classes of pairs (E,ω), where ω is a nonzero differential on E.

Suppose $F : \mathcal{R} \to \mathbf{C}$ is homogeneous of degree $-k$, so $F(\lambda L) = \lambda^{-k}F(L)$. Then we may also view T_ℓ as a sum over lattices that contain L with index ℓ, as follows. Suppose $L' \subset L$ is a sublattice of index ℓ and set $L'' = \ell^{-1}L'$. Then we have a chain of inclusions

$$\ell L \subset L' \subset L \subset \ell^{-1}L' = L''.$$

Since $[\ell^{-1}L' : L'] = \ell^2$ and $[L : L'] = \ell$, it follows that $[L'' : L] = \ell$. By homogeneity,

$$T_\ell(F)(L) = \ell^{k-1} \sum_{[L:L']=\ell} F(L') = \frac{1}{\ell} \sum_{[L'':L]=\ell} F(L''). \tag{1.4.1}$$

1.5 Hecke Operators Directly on q-expansions

Recall that the nth Hecke operator T_n of weight k is

$$T_n(L) = n^{k-1} \sum_{\substack{L' \subset L \\ [L:L']=n}} L'.$$

Modular forms of weight k correspond to holomorphic functions on lattices of degree $-k$, and T_n extend to an operator on these functions on lattices, so T_n defines on operator on $M_k(1)$. Recall that Fourier expansion defines an injective map $M_k(1) \subset \mathbf{C}[[q]]$. In this section, we describe $T_n(\sum a_n q^n)$ explicitly as a q-expansion.

1.5.1 Explicit Description of Sublattices

In order to describe T_n more explicitly, we explicitly enumerate the sublattices $L' \subset L$ of index n. More precisely, we give a basis for each L' in terms of a basis for L. Note that L/L' is a group of order n and

$$L'/nL \subset L/nL = (\mathbf{Z}/n\mathbf{Z})^2.$$

Write $L = \mathbf{Z}\omega_1 + \mathbf{Z}\omega_2$, let Y_2 be the cyclic subgroup of L/L' generated by ω_2 and let $d = \#Y_2$. If $Y_1 = (L/L')/Y_2$, then Y_1 is generated by the image of ω_1, so it is a cyclic group of order $a = n/d$. Our goal is to exhibit a basis of L'. Let $\omega_2' = d\omega_2 \in L'$

and use that Y_1 is generated by the image of ω_1 to write $a\omega_1 = \omega_1' - b\omega_2$ for some integer b and some $\omega_1' \in L'$. Since b is only well-defined modulo d we may assume $0 \le b \le d-1$. Thus

$$\begin{pmatrix}\omega_1'\\ \omega_2'\end{pmatrix} = \begin{pmatrix}a & b\\ 0 & d\end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}$$

and the change of basis matrix has determinant $ad = n$. Since

$$\mathbf{Z}\omega_1' + \mathbf{Z}\omega_2' \subset L' \subset L = \mathbf{Z}\omega_1 + \mathbf{Z}\omega_2$$

and $[L : \mathbf{Z}\omega_1' + \mathbf{Z}\omega_2'] = n$ (since the change of basis matrix has determinant n) and $[L : L'] = n$ we see that $L' = \mathbf{Z}\omega_1' + \mathbf{Z}\omega_2'$.

Proposition 1.5.1. *Let n be a positive integer. There is a one-to-one correspondence between sublattices $L' \subset L$ of index n and matrices $\left(\begin{smallmatrix}a & b\\ 0 & d\end{smallmatrix}\right)$ with $ad = n$ and $0 \le b \le d-1$.*

Proof. The correspondence is described above. To check that it is a bijection, we just need to show that if $\gamma = \left(\begin{smallmatrix}a & b\\ 0 & d\end{smallmatrix}\right)$ and $\gamma' = \left(\begin{smallmatrix}a' & b'\\ 0 & d'\end{smallmatrix}\right)$ are two matrices satisfying the listed conditions, and

$$\mathbf{Z}(a\omega_1 + b\omega_2) + \mathbf{Z}d\omega_2 = \mathbf{Z}(a\omega_1' + b\omega_2') + \mathbf{Z}d\omega_2',$$

then $\gamma = \gamma'$. Equivalently, if $\sigma \in \mathrm{SL}_2(\mathbf{Z})$ and $\sigma\gamma = \gamma'$, then $\sigma = 1$. To see this, we compute

$$\sigma = \gamma'\gamma^{-1} = \frac{1}{n}\begin{pmatrix}a'd & ab' - a'b\\ 0 & ad'\end{pmatrix}.$$

Since $\sigma \in \mathrm{SL}_2(\mathbf{Z})$, we have $n \mid a'd$, and $n \mid ad'$, and $aa'dd' = n^2$. If $a'd > n$, then because $aa'dd' = n^2$, we would have $ad' < n$, which would contradict the fact that $n \mid ad'$; also, $a'd < n$ is impossible since $n \mid a'd$. Thus $a'd = n$ and likewise $ad' = n$. Since $ad = n$ as well, it follows that $a' = a$ and $d' = d$, so $\sigma = \left(\begin{smallmatrix}1 & t\\ 0 & 1\end{smallmatrix}\right)$ for some $t \in \mathbf{Z}$. Then $\sigma\gamma = \left(\begin{smallmatrix}a & b+dt\\ 0 & d\end{smallmatrix}\right)$, which implies that $t = 0$, since $0 \le b \le d-1$ and $0 \le b + dt \le d-1$. □

Remark 1.5.2. As mentioned earlier, when $n = \ell$ is prime, there are $\ell+1$ sublattices of index ℓ. In general, the number of such sublattices is the sum of the positive divisors of n (exercise).

1.5.2 Hecke operators on q-expansions

Recall that if $f \in M_k(1)$, then f is a holomorphic functions on $\mathfrak{h} \cup \{\infty\}$ such that

$$f(\tau) = f\left(\frac{a\tau + b}{c\tau + d}\right)(c\tau + d)^{-k}$$

for all $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$. Using Fourier expansion we write

$$f(\tau) = \sum_{m=0}^{\infty} c_m e^{2\pi i \tau m},$$

and say f is a cusp form if $c_0 = 0$. Also, there is a bijection between modular forms f of weight k and holomorphic lattice functions $F : \mathcal{R} \to \mathbf{C}$ that satisfy $F(\lambda L) = \lambda^{-k} F(L)$; under this bijection F corresponds to $f(\tau) = F(\mathbf{Z}\tau + \mathbf{Z})$.

Now assume $f(\tau) = \sum_{m=0}^{\infty} c_m q^m$ is a modular form with corresponding lattice function F. Using the explicit description of sublattices from Section 1.5.1 above, we can describe the action of the Hecke operator T_n on the Fourier expansion of $f(\tau)$, as follows:

$$\begin{aligned}
T_n F(\mathbf{Z}\tau + \mathbf{Z}) &= n^{k-1} \sum_{\substack{a,b,d \\ ab=n \\ 0\leq b\leq d-1}} F((a\tau + b)\mathbf{Z} + d\mathbf{Z}) \\
&= n^{k-1} \sum d^{-k} F\left(\frac{a\tau + b}{d}\mathbf{Z} + \mathbf{Z}\right) \\
&= n^{k-1} \sum d^{-k} f\left(\frac{a\tau + b}{d}\right) \\
&= n^{k-1} \sum_{a,d,b,m} d^{-k} c_m e^{2\pi i \left(\frac{a\tau+b}{d}\right) m} \\
&= n^{k-1} \sum_{a,d,m} d^{1-k} c_m e^{\frac{2\pi i a m \tau}{d}} \frac{1}{d} \sum_{b=0}^{d-1} \left(e^{\frac{2\pi i m}{d}}\right)^b \\
&= n^{k-1} \sum_{\substack{ad=n \\ m'\geq 0}} d^{1-k} c_{dm'} e^{2\pi i a m' \tau} \\
&= \sum_{\substack{ad=n \\ m'\geq 0}} a^{k-1} c_{dm'} q^{am'}.
\end{aligned}$$

In the second to the last expression we let $m = dm'$ for $m' \geq 0$, then used that the sum $\frac{1}{d}\sum_{b=0}^{d-1}(e^{\frac{2\pi i m}{d}})^b$ is only nonzero if $d \mid m$.

Thus

$$T_n f(q) = \sum_{\substack{ad=n \\ m\geq 0}} a^{k-1} c_{dm} q^{am}$$

and if $\mu \geq 0$ then the coefficient of q^μ is

$$\sum_{\substack{a|n \\ a|\mu}} a^{k-1} c_{\frac{n\mu}{a^2}}.$$

(To see this, let $m = a/\mu$ and $d = n/a$ and substitute into the formula above.)

Remark 1.5.3. When $k \geq 1$ the coefficients of q^μ for all μ belong to the $\mathbf{Z}$-module generated by the c_m.

Remark 1.5.4. Setting $\mu = 0$ gives the constant coefficient of $T_n f$ which is

$$\sum_{a|n} a^{k-1} c_0 = \sigma_{k-1}(n) c_0.$$

Thus if f is a cusp form so is $T_n f$. ($T_n f$ is holomorphic since its original definition is as a finite sum of holomorphic functions.)

Remark 1.5.5. Setting $\mu = 1$ shows that the coefficient of q in $T_n f$ is $\sum_{a|1} 1^{k-1} c_n = c_n$. As an immediate corollary we have the following important result.

Corollary 1.5.6. *If f is a cusp form such that $T_n f$ has 0 as coefficient of q for all $n \geq 1$, then $f = 0$.*

When $n = p$ is prime, the action action of T_p on the q-expansion of f is given by the following formula:

$$T_p f = \sum_{\mu \geq 0} \sum_{\substack{a|n \\ a|\mu}} a^{k-1} c_{\frac{n\mu}{a^2}} q^\mu.$$

Since $n = p$ is prime, either $a = 1$ or $a = p$. When $a = 1$, $c_{p\mu}$ occurs in the coefficient of q^μ and when $a = p$, we can write $\mu = p\lambda$ and we get terms $p^{k-1} c_\lambda$ in $q^{\lambda p}$. Thus

$$T_p f = \sum_{\mu \geq 0} c_{p\mu} q^\mu + p^{k-1} \sum_{\lambda \geq 0} c_\lambda q^{p\lambda}.$$

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