1.6.1 *What are Néron Models?*

Suppose E is an elliptic curve over $\mathbf{Q}$. If Δ is the minimal discriminant of E, then E has good reduction at p for all $p \nmid \Delta$, in the sense that E extends to an abelian scheme $\mathcal{E}$ over $\mathbf{Z}_p$ (i.e., a "smooth" and "proper" group scheme). One can not ask for E to extend to an abelian scheme over $\mathbf{Z}_p$ for all $p \mid \Delta$. One can, however, ask whether there is a notion of "good" model for E at these bad primes. To quote [BLR, page 1], "It came as a surprise for arithmeticians and algebraic geometers when A. Néron, relaxing the condition of properness and concentrating on the group structure and the smoothness, discovered in the years 1961–1963 that such models exist in a canonical way."

Before formally defining Néron models, we describe what it means for a morphism $f : X \to Y$ of schemes to be smooth. A morphism $f : X \to Y$ is finite type if for every open affine $U = \mathrm{Spec}(R) \subset Y$ there is a finite covering of $f^{-1}(U)$ by open affines $\mathrm{Spec}(S)$, such that each S is a finitely generated R-algebra.

Definition 1.6.1. A morphism $f : X \to Y$ is *smooth at* $x \in X$ if it is of finite type and there are open affine neighborhoods $\mathrm{Spec}(A) \subset X$ of x and $\mathrm{Spec}(R) \subset Y$ of $f(x)$ such that

$$A \cong R[t_1, \ldots, t_{n+r}]/(f_1, \ldots, f_n)$$

for elements $f_1, \ldots, f_n \in R[t_1, \ldots, t_{n+r}]$ and all $n \times n$ minors of the Jacobian matrix $(\partial f_i/\partial t_j)$ generate the unit ideal of A. The morphism f is *étale* at x if, in addition, $r = 0$. A morphism is *smooth of relative dimension* d if it is smooth at x for every $x \in X$ and $r = d$ in the isomorphism above.

Smooth morphisms behave well. For example, if f and g are smooth and $f \circ g$ is defined, then $f \circ g$ is automatically smooth. Also, smooth morphisms are closed under base extension: if $f : X \to Y$ is a smooth morphism over S, and S' is a scheme over S, then the induced map $X \times_S S' \to Y \times_S S'$ is smooth. (If you've never seen products of schemes, it might be helpful to know that $\mathrm{Spec}(A) \times \mathrm{Spec}(B) = \mathrm{Spec}(A \otimes B)$. Read [4, §II.3] for more information about fiber products, which provide a geometric way to think about tensor products. Also, we often write $X_{S'}$ as shorthand for $X \times_S S'$.)

We are now ready for the definition. Suppose R is a Dedekind domain with field of fractions K (e.g., $R = \mathbf{Z}$ and $K = \mathbf{Q}$).

Definition 1.6.2 (Néron model). Let A be an abelian variety over K. The *Néron model* $\mathcal{A}$ of A is a smooth commutative group scheme over R such that for any smooth morphism $S \to R$ the natural map of abelian groups

$$\mathrm{Hom}_R(S, \mathcal{A}) \to \mathrm{Hom}_K(S \times_R K, A)$$

is a bijection. This is called the Néron mapping property: In more compact notation, it says that there is an isomorphism $\mathcal{A}(S) \cong A(S_K)$.

Taking $S = \mathcal{A}$ in the definition we see that $\mathcal{A}$ is unique, up to a unique isomorphism.

It is a deep theorem that Néron models exist. Fortunately, Bosch, Lütkebohmert, and Raynaud devoted much time to create a carefully written book [1] that explains the construction in modern language. Also, in the case of elliptic curves, Silverman's second book [9] is extremely helpful.

The basic idea of the construction is to first observe that if we can construct a Néron model at each localization $R_\mathfrak{p}$ at a nonzero prime ideal of R, then each of these local models can be glued to obtain a global Néron model (this uses that there are only finitely many primes of bad reduction). Thus we may assume that R is a discrete valuation ring.

The next step is to pass to the "strict henselization" R' of R. A local ring R with maximal ideal $\wp$ is henselian if "every simple root lifts uniquely"; more precisely, if whenever $f(x) \in R[x]$ and $\alpha \in R$ is such that $f(\alpha) \equiv 0 \pmod{\wp}$ and $f'(\alpha) \not\equiv 0 \pmod{\wp}$, there is a unique element $\tilde{\alpha} \in R$ such that $\tilde{\alpha} \equiv \alpha \pmod{\wp}$ and $f(\tilde{\alpha}) = 0$. The strict henselization of a discrete valuation ring R is an extension of R that is henselian and for which the residue field of R' is the separable closure of the residue field of R (when the residue field is finite, the separable close is just the algebraic closure). The strict henselization is not too much bigger than R, though it is typically not finitely generated over R. It is, however, much smaller than the completion of R (e.g., $\mathbf{Z}_p$ is uncountable). The main geometric property of a strictly henselian ring R with residue field k is that if X is a smooth scheme over R, then the reduction map $X(R) \to X(k)$ is surjective.

Working over the strict henselization, we first resolve singularities. Then we use a generalization of the theorem that Weil used to construct Jacobians to pass from a birational group law to an actual group law. We thus obtain the Néron model over the strict henselization of R. Finally, we use Grothendieck's faithfully flat descent to obtain a Néron model over R.

When A is the Jacobian of a curve X, there is an alternative approach that involves the "minimal proper regular model" of X. For example, when A is an elliptic curve, it is the Jacobian of itself, and the Néron model can be constructed in terms of the minimal proper regular model $\mathcal{X}$ of A as follows. In general, the model $\mathcal{X} \to R$ is not also smooth. Let $\mathcal{X}'$ be the smooth locus of $\mathcal{X} \to R$, which is obtained by removing from each closed fiber $\mathcal{X}_{\mathbf{F}_p} = \sum n_i C_i$ all irreducible components with multiplicity $n_i \geq 2$ and all singular points on each C_i, and all points where at least two C_i intersect each other. Then the group structure on A extends to a group structure on $\mathcal{X}'$, and $\mathcal{X}'$ equipped with this group structure is the Néron model of A.

Explicit determination of the possibilities for the minimal proper regular model of an elliptic curve was carried out by Kodaira, then Néron, and finally in a very explicit form by Tate. Tate codified a way to find the model in what's called "Tate's Algorithm" (see Antwerp IV, which is available on my web page: `http://modular.fas.harvard.edu/scans/antwerp/`, and look at Silverman, chapter IV, which also has important implementation advice).

1.6.2 The Birch and Swinnerton-Dyer Conjecture and Néron Models

Throughout this section, let A be an abelian variety over $\mathbf{Q}$ and let $\mathcal{A}$ be the corresponding Néron model over $\mathbf{Z}$. We work over $\mathbf{Q}$ for simplicity, but could work over any number field.

Let $L(A, s)$ be the Hasse-Weil L-function of A (see Section [to be written]). Let $r = \mathrm{ord}_{s=1} L(A, s)$ be the analytic rank of A. The Birch and Swinnerton-Dyer Conjecture asserts that $A(\mathbf{Q}) \approx \mathbf{Z}^r \oplus A(\mathbf{Q})_{\mathrm{tor}}$ and

$$\frac{L^{(r)}(A,1)}{r!} = \frac{(\prod c_p) \cdot \Omega_A \cdot \mathrm{Reg}_A \cdot \#\text{Ш}(A)}{\#A(\mathbf{Q})_{\mathrm{tor}} \cdot \#A^\vee(\mathbf{Q})_{\mathrm{tor}}}.$$

We have not defined most of the quantities appearing in this formula. In this section, we will define the Tamagawa numbers c_p, the real volume Ω_A, and the Shafarevich-Tate group $Ш(A)$ in terms of the Néron model $\mathcal{A}$ of A.

We first define the Tamagawa numbers c_p, which are the orders groups of connected components. Let p be a prime and consider the closed fiber $\mathcal{A}_{\mathbf{F}_p}$, which is a smooth commutative group scheme over $\mathbf{F}_p$. Then $\mathcal{A}_{\mathbf{F}_p}$ is a disjoint union of one or more connected components. The connected component $\mathcal{A}^0_{\mathbf{F}_p}$ that contains the identity element is a subgroup of $\mathcal{A}_{\mathbf{F}_p}$ (Intuition: the group law is continuous and the continuous image of a connected set is connected, so the group structure restricts to $\mathcal{A}^0_{\mathbf{F}_p}$).

Definition 1.6.3 (Component Group). The *component group* of A at p is

$$\Phi_{A,p} = \mathcal{A}_{\mathbf{F}_p}/\mathcal{A}^0_{\mathbf{F}_p}.$$

Fact: The component group $\Phi_{A,p}$ is a finite flat group scheme over $\mathbf{F}_p$, and for all but finitely many primes p, we have $\Phi_{A,p} = 0$.

Definition 1.6.4 (Tamagawa Numbers). The *Tamagawa number* of A at a prime p is

$$c_p = \#\Phi_{A,p}(\mathbf{F}_p).$$

Next we define the real volume Ω_A. Choose a basis

$$\omega_1, \ldots, \omega_d \in \mathrm{H}^0(\mathcal{A}, \Omega^1_{\mathcal{A}/\mathbf{Z}})$$

for the global differential 1-forms on $\mathcal{A}$, where $d = \dim A$. The wedge product $w = \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_d$ is a global d-form on $\mathcal{A}$. Then w induces a differential d-form on the real Lie group $A(\mathbf{R})$.

Definition 1.6.5 (Real Volume). The *real volume* of A is

$$\Omega_A = \left| \int_{A(\mathbf{R})} w \right| \in \mathbf{R}_{>0}.$$

Finally, we give a definition of the Shafarevich-Tate group in terms of the Néron model. Let $\mathcal{A}_0$ be the scheme obtained from the Néron model $\mathcal{A}$ over A by removing from each closed fiber all nonidentity components. Then $\mathcal{A}_0$ is again a smooth commutative group scheme, but it need not have the Néron mapping property.

Recall that an étale morphism is a morphism that is smooth of relative dimension 0. A sheaf of abelian groups on the étale site $\mathbf{Z}_{\text{ét}}$ is a functor (satisfying certain axioms) from the category of étale morphism $X \to \mathbf{Z}$ to the category of abelian groups. There are enough sheaves on $\mathbf{Z}_{\text{ét}}$ so that there is a cohomology theory for such sheaves, which is called étale cohomology. In particular if $\mathcal{F}$ is a sheaf on $\mathbf{Z}_{\text{ét}}$, then for every integer q there is an abelian group $\mathrm{H}^q(\mathbf{Z}_{\text{ét}}, \mathcal{F})$ associated to $\mathcal{F}$ that has the standard properties of a cohomology functor.

The group schemes $\mathcal{A}_0$ and $\mathcal{A}$ both determine sheaves on the étale site, which we will also denote by $\mathcal{A}_0$ and $\mathcal{A}$.

Definition 1.6.6 (Shafarevich-Tate Group). Suppose $A(\mathbf{R})$ is connected that that $\mathcal{A}_0 = \mathcal{A}$. Then the *Shafarevich-Tate* group of A is $\mathrm{H}^1(\mathbf{Z}_{\text{ét}}, \mathcal{A})$. More generally, suppose only that $A(\mathbf{R})$ is connected. Then the Shafarevich-Tate group is the image of the natural map

$$f : \mathrm{H}^1(\mathbf{Z}_{\text{ét}}, \mathcal{A}_0) \to \mathrm{H}^1(\mathbf{Z}_{\text{ét}}, \mathcal{A}).$$

Even more generally, if $A(\mathbf{R})$ is not connected, then there is a natural map $r : \mathrm{H}^1(\mathbf{Z}_{\text{ét}}, \mathcal{A}) \to \mathrm{H}^1(\mathrm{Gal}(\mathbf{C}/\mathbf{R}), A(\mathbf{C}))$ and $Ш(A) = \mathrm{im}(f) \cap \ker(r)$.

Mazur proves in the appendix to [5] that this definition is equivalent to the usual Galois cohomology definition. To do this, he considers the exact sequence $0 \to \mathcal{A}_0 \to \mathcal{A} \to \Phi_A \to 0$, where Φ_A is a sheaf version of $\oplus_p \Phi_{A,p}$. The main input is Lang's Theorem, which implies that over a local field, unramified Galois cohomology is the same as the cohomology of the corresponding component group.

Conjecture 1.6.7 (Shafarevich-Tate). *The group* $\mathrm{H}^1(\mathbf{Z}_{\text{ét}}, \mathcal{A})$ *is finite.*

When A has rank 0, all component groups $\Phi_{A,p}$ are trivial, $A(\mathbf{R})$ is connected, and $A(\mathbf{Q})_{\text{tor}}$ and $A^\vee(\mathbf{Q})_{\text{tor}}$ are trivial, the Birch and Swinnerton-Dyer conjecture takes the simple form

$$\frac{L(A,1)}{\Omega_A} = \#\,\mathrm{H}^1(\mathbf{Z}_{\text{ét}}, \mathcal{A}).$$

Later, when A is modular, we will (almost) interpret $L(A,1)/\Omega_A$ as the order of a certain group that involves modular symbols. Thus the BSD conjecture asserts that two groups have the same order; however, they are not isomorphic, since, e.g., when $\dim A = 1$ the modular symbols group is always cyclic, but the Shafarevich-Tate group is never cyclic (unless it is trivial).

1.6.3 Functorial Properties of Neron Models

The definition of Néron model is functorial, so one might expect the formation of Néron models to have good functorial properties. Unfortunately, it doesn't.

Proposition 1.6.8. *Let A and B be abelian varieties. If $\mathcal{A}$ and $\mathcal{B}$ are the Néron models of A and B, respectively, then the Néron model of $A \times B$ is $\mathcal{A} \times \mathcal{B}$.*

Suppose $R \subset R'$ is a finite extension of discrete valuation rings with fields of fractions $K \subset K'$. Sometimes, given an abelian variety A over a field K, it is easier to understand properties of the abelian variety, such as reduction, over K'. For example, you might have extra information that implies that $A_{K'}$ decomposes as a product of well-understood abelian varieties. It would thus be useful if the Néron model of $A_{K'}$ were simply the base extension $\mathcal{A}_{R'}$ of the Néron model of A over R. This is, however, frequently not the case.

Distinguishing various types of ramification will be useful in explaining how Néron models behave with respect to base change, so we now recall the notions of tame and wild ramification. If π generates the maximal ideal of R and v' is the valuation on R', then the extension is *unramified* if $v'(\pi) = 1$. It is *tamely ramified* if $v'(\pi)$ is not divisible by the residue characteristic of R, and it is *wildly ramified* if $v'(\pi)$ is divisible by the residue characteristic of R. For example, the extension $\mathbf{Q}_p(p^{1/p})$ of $\mathbf{Q}_p$ is wildly ramified.

Example 1.6.9. If R is the ring of integers of a p-adic field, then for every integer n there is a unique unramified extension of R of degree n. See [2, §I.7], where Fröhlich uses Hensel's lemma to show that the unramified extensions of $K = \mathrm{Frac}(R)$ are in bijection with the finite (separable) extensions of the residue class field.

The Néron model does not behave well with respect to base change, except in some special cases. For example, suppose A is an abelian variety over the field of fractions K of a discrete valuation ring R. If K' is the field of fractions of a finite unramified extension R' of R, then the Néron model of $A_{K'}$ is $\mathcal{A}_{R'}$, where $\mathcal{A}$ is the Néron model of A over R. Thus the Néron model over an unramified extension is obtained by base extending the Néron model over the base. This is not too surprising because in the construction of Néron model we first passed to the strict henselization of R, which is a limit of unramified extensions.

Continuing with the above notation, if K' is tamely ramified over K, then in general $\mathcal{A}_{R'}$ need *not* be the Néron model of $A_{K'}$. Assume that K' is Galois over K. In [3], Bas Edixhoven describes the Néron model of A_K in terms of $\mathcal{A}_{R'}$. To describe his main theorem, we introduce the restriction of scalars of a scheme.

Definition 1.6.10 (Restriction of Scalars). Let $S' \to S$ be a morphism of schemes and let X' be a scheme over S'. Consider the functor

$$\mathcal{R}(T) = \mathrm{Hom}_{S'}(T \times_S S', X')$$

on the category of all schemes T over S. If this functor is representable, the representing object $X = \mathrm{Res}_{S'/S}(X')$ is called the *restriction of scalars* of X' to S.

Edixhoven's main theorem is that if G is the Galois group of K' over K and $X = \mathrm{Res}_{R'/R}(\mathcal{A}_{R'})$ is the restriction of scalars of $\mathcal{A}_{R'}$ down to R, then there is a natural map $\mathcal{A} \to X$ whose image is the closed subscheme X^G of fixed elements.

We finish this section with some cautious remarks about exactness properties of Néron models. If $0 \to A \to B \to C \to 0$ is an exact sequence of abelian varieties, then the functorial definition of Néron models produces a complex of Néron models

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0,$$

where $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are the Néron models of A, B, and C, respectively. This complex can fail to be exact at every point. For an in-depth discussion of conditions when we have exactness, along with examples that violate exactness, see [1, Ch. 7], which says: "we will see that, except for quite special cases, there will be a defect of exactness, the defect of right exactness being much more serious than the one of left exactness."

To give examples in which right exactness fails, it suffices to give an optimal quotient $B \to C$ such that for some p the induced map $\Phi_{B,p} \to \Phi_{C,p}$ on component groups is not surjective (recall that optimal means $A = \ker(B \to C)$ is an abelian variety). Such quotients, with B and C modular, arise naturally in the context of Ribet's level optimization. For example, the elliptic curve E given by $y^2 + xy = x^3 + x^2 - 11x$ is the optimal new quotient of the Jacobian $J_0(33)$ of $X_0(33)$. The component group of E at 3 has order 6, since E has semistable reduction at 3 (since $3 \,||\, 33$) and $\mathrm{ord}_3(j(E)) = -6$. The image of the component group of $J_0(33)$ in the component group of E has order 2:

```
> OrderOfImageOfComponentGroupOfJ0N(ModularSymbols("33A"),3);
2
```

Note that the modular form associated to E is congruent modulo 3 to the form corresponding to $J_0(11)$, which illustrates the connection with level optimization.

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