1.5.4 Every abelian variety is a quotient of a Jacobian

Over an infinite field, *every* abelin variety can be obtained as a quotient of a Jacobian variety. The modular abelian varieties that we will encounter later are, by definition, exactly the quotients of the Jacobian $J_1(N)$ of $X_1(N)$ for some N. In this section we see that merely being a quotient of a Jacobian does not endow an abelian variety with any special properties.

Theorem 1.5.4 (Matsusaka). *Let A be an abelian variety over an algebraically closed field. Then there is a Jacobian J and a surjective map $J \to A$.*

This was originally proved in *On a generating curve of an abelian variety*, Nat. Sc. Rep. Ochanomizu Univ. **3** (1952), 1–4. Here is the Math Review by P. Samuel:

> An abelian variety A is said to be generated by a variety V (and a mapping f of V into A) if A is the group generated by $f(V)$. It is proved that every abelian variety A may be generated by a curve defined over the algebraic closure of $\mathrm{def}(A)$. A first lemma shows that, if a variety V is the carrier of an algebraic system $(X(M))_{M \in U}$ of curves $(X(M)$ being defined, non-singular and disjoint from the singular bunch of V for almost all M in the parametrizing variety U) if this system has a simple base point on V, and if a mapping f of V into an abelian variety is constant on some $X(M_0)$, then f is a constant; this is proved by specializing on M_0 a generic point M of U and by using specializations of cycles [Matsusaka, Mem. Coll. Sci. Kyoto Univ. Ser. A. Math. 26, 167–173 (1951); these Rev. 13, 379]. Another lemma notices that, for a normal projective variety V, a suitable linear family of plane sections of V may be taken as a family $(X(M))$. Then the main result follows from the complete reducibility theorem. This result is said to be the basic tool for generalizing Chow's theorem ("the Jacobian variety of a curve defined over k is an abelian projective variety defined over k").

Milne [2, §10] proves the theorem under the weaker hypothesis that the base field is infinite. We briefly sketch his proof now. If $\dim A = 1$, then A is the Jacobian of itself, so we may assume $\dim A > 1$. Embed A into $\mathbf{P}^n$, then, using the Bertini theorem, cut $A \subset \mathbf{P}^n$ by hyperplane sections $\dim(A) - 1$ times to obtain a nonsingular curve C on A of the form $A \cap V$, where V is a linear subspace of $\mathbf{P}^n$. Using standard arguments from Hartshorne [1], Milne shows (Lemma 10.3) that if W is a nonsingular variety and $\pi : W \to A$ is a finite morphism, then $\pi^{-1}(C)$ is geometrically connected (the main point is that the pullback of an ample invertible sheaf by a finite morphism is ample). (A morphism $f : X \to Y$ is *finite* if for every open affine subset $U = \mathrm{Spec}(R) \subset Y$, the inverse image $f^{-1}(U) \subset X$ is an affine open subset $\mathrm{Spec}(B)$ with B a finitely generated R-**module**. Finite morphisms have finite fibers, but not conversely.) We assume this lemma and deduce the theorem.

Let J be the Jacobian of C; by the universal property of Jacobians there is a unique homomorphism $f : J \to A$ coming from the inclusion $C \hookrightarrow A$. The image $A_1 = f(J)$ is an abelian subvariety since images of homomorphisms of abelian varieties are abelian varieties. By the Poincare reducibility theorem (we only proved this over $\mathbf{C}$, but it is true in general), there is an abelian subvariety $A_2 \subset A$ such that $A_1 + A_2 = A$ and $A_1 \cap A_2$ is finite. The isogeny $g : A_1 \times A_2 \to A$ given by $g(x, y) = x + y \in A$ is a finite morphism (any isogeny of abelian varieties is finite, flat, and surjective by Section 8 of [2]). The inverse image $g^{-1}(A_1)$ is a

union of $\#(A_1 \cap A_2)$ irreducible components; if this intersection is nontrivial, then likewise $g^{-1}(C)$ is reducible, which is a contradiction. This does not complete the proof, since it is possible that g is an isomorphism, so we use one additional trick. Suppose n is a positive integer coprime to the residue characteristic, and let

$$h = 1 \times [n] : A_1 \times A_2 \to A_1 \times A_2$$

be the identity map on the first factor and multiplication by n on the second. Then h is finite and $(h \circ g)^{-1}(A_1)$ is a union of $n^{2 \dim A_2} = \deg(h)$ irreducible components, hence $(h \circ g)^{-1}(C)$ is reducible, a contradiction.

Question 1.5.5. Is Theorem 1.5.4 false for some abelian variety A over some finite field k?

Question 1.5.6 (Milne). Using the theorem we can obtain a sequence of Jacobian varieties $J_1, J_2, \ldots$ that form a complex

$$\ldots \to J_2 \to J_1 \to A \to 0.$$

(In each case the image of J_{i+1} is the connected component of the kernel of $J_i \to J_{i-1}$.) Is it possible to make this construction in such a way that the sequence terminates in 0?

Question 1.5.7 (Yau). Let A be an abelian variety. What can be said about the minimum of the dimensions of all Jacobians J such that there is a surjective morphism $J \to A$?

Remark 1.5.8. Brian Conrad has explained to the author that if A is an abelian variety over an infinite field, then A can be embedded in a Jacobian J. This does not follow directly from Theorem 1.5.4 above, since if $J \twoheadrightarrow A^\vee$, then the dual map $A \to J$ need not be injective.

1.6   Néron Models

The main references for Néron models are as follows:

1. [AEC2]: Silverman, *Advanced Topics in the Arithmetic of Elliptic Curves*. Chapter IV of this book contains an extremely well written and motivated discussion of Néron models of elliptic curves over Dedekind domains with perfect residue field. In particular, Silverman gives an almost complete construction of Néron models of elliptic curves. Silverman very clearly really wants his reader to understand the construction. Highly recommended.

2. [BLR]: Bosch, Lütkebohmert, Raynaud, *Néron Models*. This is an excellent and accessible book that contains a complete construction of Néron models and some of their generalizations, a discussion of their functorial properties, and a sketch of the construction of Jacobians of families of curves. The goal of this book was to redo in scheme-theoretic language Néron original paper, which is written in a language that was ill-adapted to the subtleties of Néron models.

3. Artin, *Néron Models*, in Cornell-Silverman. This is the first-ever exposition of Néron's original paper in the language of schemes.

1.6.1   What are Néron Models?

Suppose E is an elliptic curve over $\mathbf{Q}$. If Δ is the minimal discriminant of E, then E has good reduction at p for all $p \nmid \Delta$, in the sense that E extends to an abelian scheme $\mathcal{E}$ over $\mathbf{Z}_p$ (i.e., a "smooth" and "proper" group scheme). One can not ask for E to extend to an abelian scheme over $\mathbf{Z}_p$ for all $p \mid \Delta$. One can, however, ask whether there is a notion of "good" model for E at these bad primes. To quote [BLR, page 1], "It came as a surprise for arithmeticians and algebraic geometers when A. Néron, relaxing the condition of properness and concentrating on the group structure and the smoothness, discovered in the years 1961–1963 that such models exist in a canonical way."

Before formally defining Néron models, we describe what it means for a morphism $f : X \to Y$ of schemes to be smooth. A morphism $f : X \to Y$ is finite type if for every open affine $U = \operatorname{Spec}(R) \subset Y$ there is a finite covering of $f^{-1}(U)$ by open affines $\operatorname{Spec}(S)$, such that each S is a finitely generated R-algebra.

Definition 1.6.1. A finite-type morphism $f : X \to Y$ is *smooth at* $x \in X$ if there are open affine neighborhoods $\operatorname{Spec}(A) \subset X$ of x and $\operatorname{Spec}(R) \subset Y$ of $f(x)$ such that

$$A \cong R[t_1, \ldots, t_{n+r}]/(f_1, \ldots, f_n)$$

for elements $f_1, \ldots, f_n \in R[t_1, \ldots, t_{n+r}]$ and all $n \times n$ minors of the Jacobian matrix $(\leq f_i/\partial t_j)$ generate the unit ideal of A. The morphism f is *étale* if, in addition, $r = 0$.

Smooth morphisms behave well. For example, if f and g are smooth and $f \circ g$ is defined then $f \circ g$ is automatically smooth. Also, smooth morphisms are closed under base extension: if $f : X \to Y$ is a smooth morphism over S, and S' is a scheme over S, then the induced map $X \times_S S' \to Y \times_S S'$ is smooth. (By the way, if you've never seen products of schemes, it'll be helpful to know that $\operatorname{Spec}(R) \times \operatorname{Spec}(S) = \operatorname{Spec}(R \otimes S)$. Read Hartshorne or ask me for more information about fiber products, which aren't too hard, and provide a nice geometric way to think about tensor products.)

We are now ready for the definition. Suppose R is a Dedekind domain with field of fractions K (e.g., $R = \mathbf{Z}$ and $K = \mathbf{Q}$).

Definition 1.6.2 (Néron model). Let A be an abelian variety over K. The *Néron model* $\mathcal{A}$ of A is a smooth commutative group scheme over R such that for any smooth $S \to \operatorname{Spec}(R)$ the natural map

$$\operatorname{Hom}(S, \mathcal{A}) \to \operatorname{Hom}(S \times_R K, A)$$

is a bijection.

Taking $S = \mathcal{A}$ in the definition we see that $\mathcal{A}$ is unique, up to a unique isomorphism.

It is a deep theorem that Néron models exist. Fortunately, Bosch, Lütkebohmert, and Raynaud devoted much time to create a carefully written book that explains the construction in modern language. Also, in the case of elliptic curves, Silverman's second book is extremely helpful.

The basic idea of the construction is to first observe that if we can construct a Néron model at each localization $R_{\mathfrak{p}}$ at a nonzero prime ideal of R, then each of these local models can be glued to obtain a global Néron model (this uses that

there are only finitely many primes of bad reduction). Thus we may assume that R is a discrete valuation ring.

The next step is to pass to the "strict henselization" R' of R. A local ring R with maximal ideal $\wp$ is henselian if "every simple root lifts uniquely"; more precisely, if whenever $f(x) \in R[x]$ and $\alpha \in R$ is such that $f(\alpha) \equiv 0 \pmod{\wp}$ and $f'(\alpha) \not\equiv 0 \pmod{\wp}$, there is a unique element $\tilde{\alpha} \in R$ such that $\tilde{\alpha} \equiv \alpha \pmod{\wp}$ and $f(\tilde{\alpha}) = 0$. The strict henselization of a discrete valuation ring R is an extension of R that is henselian and for which the residue field of R' is the separable closure of the residue field of R (when the residue field is finite, the separable close is just the algebraic closure). The strict henselization is not too much bigger than R, though it is typically not finitely generated over R. It is, however, much smaller than the completion of R (e.g., $\mathbf{Z}_p$ is uncountable). The main geometric property of a strictly henselian ring R with residue field k is that if X is a smooth scheme over R, then the reduction map $X(R) \to X(k)$ is surjective.

Working over the strict henselization, we resolve singularities and use a generalization of the theorem that Weil used to construct Jacobians by passing from a birational group law to an actual group law on a related object to obtain the Néron model of the strict henselization. Finally, we use Grothendieck's faithfully flat descent to obtain a Néron model over the original discrete valuation ring.

When A is the Jacobian of a curve there is an alternative approach. For example, when A is an elliptic curve, it is the Jacobian of itself, and the Néron model can be constructed in terms of the "minimal proper regular" model $\mathcal{X}$ of A as follows. In general, $\mathcal{X} \to R$ is not be smooth. Let $\mathcal{X}'$ be the smooth locus of $\mathcal{X} \to R$, which is obtained by removing from each closed fiber $\mathcal{X}_{\mathbf{F}_p} = \sum n_i C_i$ all irreducible components with multiplicity $n_i \geq 2$ and all singular points on each C_i, and all points where at least two C_i intersect each other. Then the group structure on A extends to a group structure on $\mathcal{X}'$, and $\mathcal{X}'$ equipped with this group structure is the Néron model of A.

Explicit determination of the possibilities for the minimal proper regular model of an elliptic curve was carried out by Kodaira, then Néron, and finally in a very explicit form by Tate. Tate codified a way to find the model in what's called "Tate's Algorithm" (see Antwerp IV, which is available on my web page: `http://modular.fas.harvard.edu/scans/antwerp/`, and look at Silverman, chapter IV, which also has important implementation advice).

Néron models exist. In the next sections, we'll see some ways in which they will be directly relevant to this course.

1.6.2   Some Motivation: The Birch and Swinnerton-Dyer Conjecture

The Real Volume

Étale Cohomology and the Shafarevich-Tate Group

Tamagawa Numbers

1.6.3   Functorial Properties of Neron Models

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