1.5 Jacobians of Curves

1. I will give a research talk in the modular curves seminar tomorrow (Thursday, October 16, 3-4pm in SC 507) about the arithmetic of the Jacobian $J_1(p)$ of $X_1(p)$. It will fit well with this part of the course.

2. New web page: `http://modular.fas.harvard.edu/calc/`. Type or paste in a MAGMA or PARI program, click a button, get the output. No need to install MAGMA or PARI or log in anywhere.

Today we're going to learn about Jacobians. First, some inspiring words by David Mumford:

> "The Jacobian has always been a corner-stone in the analysis of algebraic curves and compact Riemann surfaces. [...] Weil's construction [of the Jacobian] was the basis of his epoch-making proof of the Riemann Hypothesis for curves over finite fields, which really put characteristic p algebraic geometry on its feet." – Mumford, *Curves and Their Jacobians*, page 49.

1.5.1 Divisors on Curves and Linear Equivalence

Let X be a projective nonsingular algebraic curve over an algebraically field k. A *divisor* on X is a formal finite $\mathbf{Z}$-linear combination $\sum_{i=1}^{m} n_i P_i$ of closed points in X. Let $\mathrm{Div}(X)$ be the group of all divisors on X. The *degree* of a divisor $\sum_{i=1}^{m} n_i P_i$ is the integer $\sum_{i=1}^{m} n_i$. Let $\mathrm{Div}^0(X)$ denote the subgroup of divisors of degree 0.

Suppose k is a perfect field (for example, k has characteristic 0 or k is finite), but do not require that k be algebraically closed. Let the group of divisors on X over k be the subgroup

$$\mathrm{Div}(X) = \mathrm{Div}(X/k) = \mathrm{H}^0(\mathrm{Gal}(\overline{k}/k), \mathrm{Div}(X/\overline{k}))$$

of elements of $\mathrm{Div}(X/\overline{k})$ that are fixed by every automorphism of $\overline{k}/k$. Likewise, let $\mathrm{Div}^0(X/k)$ be the elements of $\mathrm{Div}(X/k)$ of degree 0.

A *rational function* on an algebraic curve X is a function $X \to \mathbf{P}^1$, defined by polynomials, which has only a finite number of poles. For example, if X is the elliptic curve over k defined by $y^2 = x^3 + ax + b$, then the field of rational functions on X is the fraction field of the integral domain $k[x, y]/(y^2 - (x^3 + ax + b))$. Let $K(X)$ denote the field of all rational functions on X defined over k.

There is a natural homomorphism $K(X)^* \to \mathrm{Div}(X)$ that associates to a rational function f its divisor

$$(f) = \sum \mathrm{ord}_P(f) \cdot P$$

where $\mathrm{ord}_P(f)$ is the order of vanishing of f at P. Since X is nonsingular, the local ring of X at a point P is isomorphic to $k[[t]]$. Thus we can write $f = t^r g(t)$ for some unit $g(t) \in k[[t]]$. Then $R = \mathrm{ord}_P(f)$.

Example 1.5.1. If $X = \mathbf{P}^1$, then the function $f = x$ has divisor $(0) - (\infty)$. If X is the elliptic curve defined by $y^2 = x^3 + ax + b$, then

$$(x) = (0, \sqrt{b}) + (0, -\sqrt{b}) - 2\infty,$$

and

$$(y) = (x_1, 0) + (x_2, 0) + (x_3, 0) - 3\infty,$$

where x_1, x_2, and x_3 are the roots of $x^3+ax+b=0$. A uniformizing parameter t at the point ∞ is x/y. An equation for the elliptic curve in an affine neighborhood of ∞ is $Z = X^3 + aXZ^2 + bZ^3$ (where $\infty = (0,0)$ with respect to these coordinates) and $x/y = X$ in these new coordinates. By repeatedly substituting Z into this equation we see that Z can be written in terms of X.

It is a standard fact in the theory of algebraic curves that if f is a nonzero rational function, then $(f) \in \mathrm{Div}^0(X)$, i.e., the number of poles of f equals the number of zeros of f. For example, if X is the Riemann sphere and f is a polynomial, then the number of zeros of f (counted with multiplicity) equals the degree of f, which equals the order of the pole of f at infinity.

The *Picard group* $\mathrm{Pic}(X)$ of X is the group of divisors on X modulo linear equivalence. Since divisors of functions have degree 0, the subgroup $\mathrm{Pic}^0(X)$ of divisors on X of degree 0, modulo linear equivalence, is well defined. Moreover, we have an exact sequence of abelian groups

$$0 \to K(X)^* \to \mathrm{Div}^0(X) \to \mathrm{Pic}^0(X) \to 0.$$

Thus for any algebraic curve X we have associated to it an abelian group $\mathrm{Pic}^0(X)$. Suppose $\pi : X \to Y$ is a morphism of algebraic curves. If D is a divisor on Y, the pullback $\pi^*(D)$ is a divisor on X, which is defined as follows. If $P \in \mathrm{Div}(Y/\overline{k})$ is a point, let $\pi^*(P)$ be the sum $\sum e_{Q/P} Q$ where $\pi(Q) = P$ and $e_{Q/P}$ is the ramification degree of Q/P. (Remark: If t is a uniformizer at P then $e_{Q/P} = \mathrm{ord}_Q(\phi^* t_P)$.) One can show that $\pi^* : \mathrm{Div}(Y) \to \mathrm{Div}(X)$ induces a homomorphism $\mathrm{Pic}^0(Y) \to \mathrm{Pic}^0(X)$. Furthermore, we obtain the contravariant *Picard functor* from the category of algebraic curves over a fixed base field to the category of abelian groups, which sends X to $\mathrm{Pic}^0(X)$ and $\pi : X \to Y$ to $\pi^* : \mathrm{Pic}^0(Y) \to \mathrm{Pic}^0(X)$.

Alternatively, instead of defining morphisms by pullback of divisors, we could consider the push forward. Suppose $\pi : X \to Y$ is a morphism of algebraic curves and D is a divisor on X. If $P \in \mathrm{Div}(X/\overline{k})$ is a point, let $\pi_*(P) = \pi(P)$. Then π_* induces a morphism $\mathrm{Pic}^0(X) \to \mathrm{Pic}^0(Y)$. We again obtain a functor, called the covariant *Albanese functor* from the category of algebraic curves to the category of abelian groups, which sends X to $\mathrm{Pic}^0(X)$ and $\pi : X \to Y$ to $\pi_* : \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(Y)$.

1.5.2 Algebraic Definition of the Jacobian

First we describe some universal properties of the Jacobian under the hypothesis that $X(k) \neq \emptyset$. Thus suppose X is an algebraic curve over a field k and that $X(k) \neq \emptyset$. The Jacobian variety of X is an abelian variety J such that for an extension k'/k, there is a (functorial) isomorphism $J(k') \to \mathrm{Pic}^0(X/k')$. (I don't know whether this condition uniquely characterizes the Jacobian.)

Fix a point $P \in X(k)$. Then we obtain a map $f : X(k) \to \mathrm{Pic}^0(X/k)$ by sending $Q \in X(k)$ to the divisor class of $Q-P$. One can show that this map is induced by an injective morphism of algebraic varieties $X \hookrightarrow J$. This morphism has the following universal property: if A is an abelian variety and $g : X \to A$ is a morphism that sends P to $0 \in A$, then there is a unique homomorphism $\psi : J \to A$ of abelian varieties such that $g = \psi \circ f$:

$$\begin{array}{ccc} X & \xrightarrow{f} & J \\ & \searrow^{g} & \downarrow \psi \\ & & A \end{array}$$

This condition uniquely characterizes J, since if $f' : X \to J'$ and J' has the universal property, then there are unique maps $J \to J'$ and $J' \to J$ whose composition in both directions must be the identity (use the universal property with $A = J$ and $f = g$).

If X is an arbitrary curve over an arbitrary field, the Jacobian is an abelian variety that represents the "sheafification" of the "relative Picard functor". Look in Milne's article or Bosch-Lüktebohmert-Raynaud *Neron Models* for more details. Knowing this totally general definition won't be important for this course, since we will only consider Jacobians of modular curves, and these curves always have a rational point, so the above properties will be sufficient.

A useful property of Jacobians is that they are canonically principally polarized, by a polarization that arises from the "θ divisor" on J. In particular, there is always an isomorphism $J \to J^\vee = \mathrm{Pic}^0(J)$.

1.5.3 The Abel-Jacobi Theorem

Over the complex numbers, the construction of the Jacobian is classical. It was first considered in the 19th century in order to obtain relations between integrals of rational functions over algebraic curves (see Mumford's book, *Curves and Their Jacobians*, Ch. III, for a nice discussion).

Let X be a Riemann surface, so X is a one-dimensional complex manifold. Thus there is a system of coordinate charts (U_α, t_α), where $t_\alpha : U_\alpha \to \mathbf{C}$ is a homeomorphism of U_α onto an open subset of $\mathbf{C}$, such that the change of coordinate maps are analytic isomorphisms. A *differential* 1*-form* on X is a choice of two continuous functions f and g to each local coordinate $z = x + iy$ on $U_\alpha \subset X$ such that $f\,dx + g\,dy$ is invariant under change of coordinates (i.e., if another local coordinate patch U'_α intersects U_α, then the differential is unchanged by the change of coordinate map on the overlap). If $\gamma : [0,1] \to X$ is a path and $\omega = f\,dx + g\,dy$ is a 1-form, then

$$\int_\gamma \omega := \int_0^1 \left(f(x(t), y(t)) \frac{dx}{dt} + g(x(t), y(t)) \frac{dy}{dt} \right) dt \in \mathbf{C}.$$

From complex analysis one sees that if γ is homologous to γ', then $\int_\gamma \omega = \int_{\gamma'} \omega$. In fact, there is a nondegenerate pairing

$$\mathrm{H}^0(X, \Omega^1_X) \times \mathrm{H}_1(X, \mathbf{Z}) \to \mathbf{C}$$

If X has genus g, then it is a standard fact that the complex vector space $\mathrm{H}^0(X, \Omega^1_X)$ of holomorphic differentials on X is of dimension g. The integration pairing defined above induces a homomorphism from integral homology to the dual V of the differentials:

$$\Phi : \mathrm{H}_1(X, \mathbf{Z}) \to V = \mathrm{Hom}(H^0(X, \Omega^1_X), \mathbf{C}).$$

This homomorphism is called the *period mapping.*

Theorem 1.5.2 (Abel-Jacobi). *The image of* Φ *is a lattice in* V.

The proof involves repeated clever application of the residue theorem.

The intersection pairing

$$\mathrm{H}_1(X,\mathbf{Z}) \times \mathrm{H}_1(X,\mathbf{Z}) \to \mathbf{Z}$$

defines a nondegenerate alternating pairing on $L = \Phi(\mathrm{H}_1(X,\mathbf{Z}))$. This pairing satisfies the conditions to induce a nondegenerate Riemann form on V, which gives $J = V/L$ to structure of abelian variety. The abelian variety J is the Jacobian of X, and if $P \in X$, then the functional $\omega \mapsto \int_P^Q \omega$ defines an embedding of X into J. Also, since the intersection pairing is perfect, it induces an isomorphism from J to $J^\vee$.

Example 1.5.3. For example, suppose $X = X_0(23)$ is the modular curve attached to the subgroup $\Gamma_0(23)$ of matrices in $\mathrm{SL}_2(\mathbf{Z})$ that are upper triangular modulo 24. Then $g = 2$, and a basis for $\mathrm{H}_1(X_0(23),\mathbf{Z})$ in terms of modular symbols is

$$\{-1/19, 0\}, \quad \{-1/17, 0\}, \quad \{-1/15, 0\}, \quad \{-1/11, 0\}.$$

The matrix for the intersection pairing on this basis is

$$\begin{pmatrix} 0 & -1 & -1 & -1 \\ 1 & 0 & -1 & -1 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

With respect to a reduced integral basis for

$$\mathrm{H}^0(X, \Omega_X^1) \cong S_2(\Gamma_0(23)),$$

the lattice $\Phi(\mathrm{H}_1(X,\mathbf{Z}))$ of periods is (approximately) spanned by

```
[
    (0.59153223605591049412844857432 - 1.68745927346801253993135357636*i
        0.762806324458047168681080323846571478727 - 0.60368764497868211035115379488*i),
    (-0.59153223605591049412844857432 - 1.68745927346801253993135357636*i
        -0.762806324458047168681080323846571478727 - 0.60368764497868211035115379488*i),
    (-1.354338560513957662809528899804 - 1.0837716284893304295801997808568748714097*i
        -0.59153223605591049412844857401 + 0.480083983510648319229045987467*i),
    (-1.52561264891609433736216065099 0.342548176804273349105263499648)
]
```

10. Jacobians of modular curves

In this section we will examine the Jacobians of modular curves, their reduction modulo primes, and the endomorphisms induced by Hecke operators.

10.1. Abelian varieties and Jacobians.

PRIMARY REFERENCES:
[**Mum1**], [**Rosen**], [**Mil2**], [**Mil3**] and [**BLRa**, Chapters 8,9].

We now review some generalities concerning abelian varieties and Jacobians of algebraic curves.

We first recall that an abelian variety A over an algebraically closed field k is a proper group variety over k. It is necessarily smooth, projective and commutative [**Mil2**, §1,2]. One can consider more generally abelian schemes, or families of abelian varieties, over an arbitrary base scheme S. An abelian scheme over S is a smooth proper group scheme over S whose geometric fibers are abelian varieties [**Mil2**, §20].

If $k = \mathbf{C}$ and A is a g-dimensional abelian variety, then the complex manifold $A(\mathbf{C})$ is isomorphic to a complex torus V/L where V is a g-dimensional vector space and L is a discrete subgroup of rank $2g$ [**Rosen**, §1]. An arbitrary complex torus V/L can be identified with the set of complex points of an abelian variety over $\mathbf{C}$ if and only if V/L possesses a non-degenerate Riemann form [**Rosen**, §3], i.e., a positive definite Hermitian form on V whose imaginary part is integer valued on L. In this case, the same is true for the complex torus V^*/L^* where $V^* \subset \mathrm{Hom}_{\mathbf{R}}(V, \mathbf{C})$ is the space of conjugate linear functions on V (i.e., additive functions ϕ satisfying $\phi(zv) = \bar{z}\phi(v)$ for all $z \in \mathbf{C}$, $v \in V$), and $L^* = \{\phi \in V^* | \phi(L) \subset \mathbf{R} + i\mathbf{Z}\}$. If A and A^* are abelian varieties satisfying $A(\mathbf{C}) \cong V/L$ and $A^*(\mathbf{C}) \cong V^*/L^*$, then A^* is called the dual abelian variety of A [**Rosen**, §4]. Note that A is isomorphic to $(A^*)^*$.

Now let C be a Riemann surface and let W denote the complex vector space of holomorphic differentials on C. Consider the complex torus V/L where $V = \mathrm{Hom}(W, \mathbf{C})$ and L is the image of the map $H_1(C, \mathbf{Z}) \to \mathrm{Hom}(W, \mathbf{C})$ defined by integration. Note that the cotangent space of V/L at the origin may be naturally identified with W. The intersection pairing on $H_1(C, \mathbf{Z})$ can be used to define a nondegenerate Riemann form on V/L, and the resulting abelian variety J is called the Jacobian of C [**Mil3**, §2]. Moreover this Riemann form gives rise to a canonical isomorphism $J \cong J^*$.

Another interpretation of the Jacobian of C is provided by the Picard functor Pic^0 (see [**Mil3**, §1]). Let $\mathrm{Div}^0(C)$ denote the group of divisors on C of degree zero, and let $\mathrm{Pic}^0(C)$ denote $\mathrm{Div}^0(C)$ modulo the group of principal divisors. Integration then defines a natural map $\mathrm{Div}^0(C) \to V/L$ which, according to the Abel-Jacobi theorem, induces a natural isomorphism of groups $\mathrm{Pic}^0(C) \cong J(\mathbf{C})$. Now choose a base-point P in C and define a mapping $C \to \mathrm{Pic}^0(C)$ by sending Q to the divisor $Q - P$. The resulting map $C \to V/L$ is analytic and induces an isomorphism $H^0(J(\mathbf{C}), \Omega^1) \to H^0(C, \Omega^1) = W$ which is independent of the base-point. Moreover the isomorphism is compatible with the natural identification of W with the cotangent space of $J(\mathbf{C}) \cong V/L$ at the origin.

To describe the Jacobian of a curve over any field, or indeed an arbitrary base scheme S, we use the Picard functor [**Mil3**, §8], [**BLRa**, Chapter 8]. For a morphism of schemes $s : X \to S$, Grothendieck [**Gro1**] defines a relative Picard

functor $\mathrm{Pic}_{X/S}$ on S-schemes by "sheafifying" the functor which sends T to the group of isomorphism classes of invertible sheaves on $X_T = X \times_S T$. Under quite general hypotheses (see Chapters 8 and 9 of [**BLRa**]) this contravariant functor is represented by a group scheme over S, and we denote its identity component $\mathrm{Pic}^0_{X/S}$. The definition is functorial in X, so that a morphism $Y \to X$ of S-schemes gives rise to a natural transformation $\mathrm{Pic}_{X/S} \to \mathrm{Pic}_{Y/S}$ and consequently a morphism $\mathrm{Pic}^0_{X/S} \to \mathrm{Pic}^0_{Y/S}$. We remark also that formation of $\mathrm{Pic}^0_{X/S}$ commutes with base change, meaning that $\mathrm{Pic}^0_{(X_T)/T}$ is naturally isomorphic to $\mathrm{Pic}^0_{X/S} \times_S T$.

If $X \to S$ is a relative curve, meaning that it is smooth and proper and its geometric fibers are curves, then $\mathrm{Pic}^0_{X/S}$ is an abelian scheme which we denote $J_{X/S}$ and call the Jacobian of X (over S), [**BLRa**, §9.2]. If also $S = \mathrm{Spec}\, k$ for an algebraically closed field k, then $\mathrm{Pic}_{X/S}(S)$ may be identified with the group of invertible sheaves on X, or equivalently, with $\mathrm{Div}(X)$ modulo the group of principal divisors. Then $\mathrm{Pic}^0_{X/S}(S)$ may be identified with $\mathrm{Pic}^0(X)$, the group $\mathrm{Div}^0(X)$ modulo the group of principal divisors. Moreover if $k = \mathbf{C}$, then the isomorphism $V/L \cong J(\mathbf{C}) \cong J_{X/\mathbf{C}}(\mathbf{C})$ is analytic, so our two descriptions of the Jacobian in this case are equivalent.

The relative Picard functor also provides a general construction of the dual of an abelian scheme. If A is an abelian scheme over S, then $\mathrm{Pic}_{A/S}$ is representable by a scheme, and $\mathrm{Pic}^0_{A/S}$ is an abelian scheme, [**BLRa**, §8.4, Theorem 5], [**FaCh**, I.1]. We write A^* for $\mathrm{Pic}^0_{A/S}$ and call it the dual abelian scheme of A. Again there is a natural isomorphism $A \cong (A^*)^*$. For a relative curve X over S there is a general construction of a "Θ-divisor" on $J_{X/S}$ which gives rise to an isomorphism $\phi_{X/S}$ of $J_{X/S}$ with $J^*_{X/S}$, [**BLRa**, §9.4]. The constructions of the dual abelian scheme, its biduality and the autoduality of the Jacobian are compatible with base-change. They are also compatible with the descriptions given above in the case $S = \mathrm{Spec}\, \mathbf{C}$.

A morphism $\pi : Y \to X$ of relative curves over S induces by Picard functoriality a homomorphism of abelian schemes $\pi^* : J_{X/S} \to J_{Y/S}$. We obtain also a homomorphism $\pi_* : J_{Y/S} \to J_{X/S}$ defined by the composite $\phi^{-1}_{Y/S} \circ (\pi^*)^* \phi_{X/S}$ where $(\pi^*)^* : J^*_{Y/S} \to J^*_{X/S}$ is again defined by Picard functoriality. We thus have two functors from the category of relative curves over S to the category of abelian schemes over S; the contravariant Picard functor Pic^0 defined by $\mathrm{Pic}^0(X) = J_{X/S}$ and $\mathrm{Pic}^0(\pi) = \pi^*$, and the covariant Albanese functor Alb defined by $\mathrm{Alb}(X) = J_{X/S}$ and $\mathrm{Alb}(\pi) = \pi_*$, [**Mil3**, §6]. If $S = \mathrm{Spec}\, k$ for an algebraically closed field k, then π^* on $J_{X/S}(S)$ is induced by the map $\mathrm{Div}(X) \to \mathrm{Div}(Y)$ defined by pull-back of divisors; a point $x \in X(S)$ is sent to $\sum_{y \in \pi^{-1}(x)} e_{y/x} y$ where $e_{y/x}$ is the ramification degree. On the other hand, π_* on $J_{Y/S}(S)$ is induced by the map $\mathrm{Div}(Y) \to \mathrm{Div}(X)$ which sends $y \in Y(S)$ to $\pi(y)$. Note that $\pi_* \circ \pi^*$ is simply multiplication by the degree of π.

There is in general a natural isomorphism of $s_*\Omega^1_{X/S}$ with the cotangent sheaf $i^*\Omega^1_{J_{X/S}/S}$ along the zero section $i : S \to J_{X/S}$. For $S = \mathrm{Spec}\, k$, this can be viewed as an isomorphism $H^0(X, \Omega^1_{X/S}) \cong \mathrm{Cot}_0(J_{X/S})$ (see [**Mil3**, Proposition 2.2]). Consider now the maps induced by π^* and π_* on the cotangent spaces at

Lecture III: How Jacobians and Theta Functions Arise

I would like to begin by introducing Jacobians in the way that they actually were discovered historically. Unfortunately, my knowledge of 19th-century literature is very scant so this should not be taken too literally. You know the story began with Abel and Jacobi investigating general algebraic integrals

$$I = \int f(x)dx$$

where f was a multi-valued algebraic function of x, i.e., the solution to

$$g(x, f(x)) \equiv 0, \quad g \text{ polynomial in } 2 \text{ variables.}$$

So we can write I as

$$I = \int_{\gamma} y \, dx$$

where γ is a path in plane curve $g(x,y) = 0$; or we may reformulate this as the study of integrals

$$I(a) = \int_{a_0}^{a} \overbrace{\frac{P(x,y)}{Q(x,y)}dx}^{\omega}, \qquad \begin{array}{l} P,Q \text{ polynomials} \\ a,a_0 \in \text{plane curve } C\colon g(x,y) = 0 \end{array}$$

of rational differentials ω on plane curves C. The main result is that such integrals always admit an addition theorem: i.e., there is an integer g such that if a_0 is a base point, and $a_1,\ldots,a_{g+1}$ are any points of C, then one can determine up to permutation $b_1, \cdots, b_g \in C$ rationally in terms of the a's* such that

$$\int_{a_0}^{a_1} \omega + \cdots + \int_{a_0}^{a_{g+1}} \omega \equiv \int_{a_0}^{b_1} \omega + \cdots + \int_{a_0}^{b_g} \omega, \quad \text{mod periods of } \int \omega.$$

For instance, if $C = \mathbb{P}^1$, $\omega = dx/x$, then $g = 1$ and:

$$\int_1^{a_1} \frac{dx}{x} + \int_1^{a_2} \frac{dx}{x} = \int_1^{a_1 a_2} \frac{dx}{x} .$$

Iterating, this implies that for all $a_1, \cdots, a_g, b_1, \cdots, b_g \in C$, there are $c_1, \cdots, c_g \in C$ depending up to permutation rationally on the a's and b's such that

$$\sum_{i=1}^{g} \int_{a_0}^{a_i} \omega + \sum_{i=1}^{g} \int_{a_0}^{b_i} \omega \equiv \sum_{i=1}^{g} \int_{a_0}^{c_i} \omega \quad \text{(mod periods)} .$$

Now this looks like a group law! Only a very slight strengthening will lead us to a reformulation in which this most classical of all theorems will suddenly sound very modern. We introduce the concept of an algebraic group G: succinctly, this is a "group object in the category of varieties," i.e., it is simultaneously a variety and a group where the group law $m: G \times G \longrightarrow G$ and the inverse $i: G \longrightarrow G$ are morphisms of varieties. Such a G is, of course, automatically a complex analytic Lie group too, hence it has a Lie algebra Lie(G), and an exponential map exp: $\mathrm{Lie}(G) \longrightarrow G$. Now I wish to rephrase

---

* E.g., one can find polynomials $g_i(x, y; a)$ in x, y and the coordinates of the a's such that the b_i's are the set of all $b \in C$ such that $g_i(b; a) = 0$.

Abel's theorem as asserting that if C is a curve, and ω is any rational differential on C, then the multi-valued function

$$a \longmapsto \int_{a_o}^{a} \omega$$

can be factored into a composition of 3 functions:

$$C\text{-(poles of } \omega) \xrightarrow{\ \phi\ } J \xleftarrow{\ \exp\ } \text{Lie } J \xrightarrow{\ \ell\ } \mathbb{C}$$

where:

i) J is a commutative algebraic group,

ii) ℓ is a linear map from Lie J to $\mathbb{C}$

iii) ϕ is a morphism of varieties; and, in fact, if $g = \dim J$, then if we use addition on J to extend ϕ to

$$\phi^{(g)}: [(C\text{-poles } \omega)\times\cdots\times(C\text{-poles } \omega)/\text{permutations}] \longrightarrow J$$
$$S_g$$

then $\phi^{(g)}$ is birational, i.e., is bijective on a Zariski-open set.

In our example

$$C = \mathbb{P}^1, \qquad \omega = dx/x,$$

then $J = \mathbb{P}^1-(0,\infty)$ which is an algebraic group where the group law is multiplication, and ϕ is the identity. The point is that J is the object that realizes the rule by which 2 g-tuples $(a_1,\cdots,a_g),(b_1,\cdots,b_g)$ are "added" to form a third $(c_1,\cdots,c_g)$, and so that the integral

$\sum_{i=1}^{g} I(x_i)$ becomes a homomorphism from J to $\mathbb{C}$. A slightly less fancy way to put it is that there is a ϕ: C-(poles ω) $\longrightarrow$ J and a <u>translation-invariant</u> differential η on J such that

$$\phi^*\eta = \omega,$$

hence

$$\int_{\phi(a_o)}^{\phi(a)} \eta \equiv \int_{a_o}^{a_o} \omega \quad \text{(mod periods)}.$$

Among the ω's, the most important are those of 1^{st} kind, i.e., without poles, and if we integrate all of them at once, we are led to the most important J of all: the <u>Jacobian</u>, which we call Jac. From property (iii), we find that Jac must be a <u>compact</u> commutative algebraic group, i.e., a complex torus, and we want that

$$\phi: \; C \longrightarrow \text{Jac},$$

should set up a bijection:

iv) $$\phi^*: \begin{bmatrix}\text{translation-}\\ \text{invariant 1-forms}\\ \eta \text{ on Jac}\end{bmatrix} \longrightarrow \begin{bmatrix}\text{rational differentials}\\ \omega \text{ on C w/o poles}\end{bmatrix} =$$

Thus

$$\dim \text{Jac} = \dim R_1(C)$$
$$= \text{genus } g \text{ of } C.$$

To construct Jac explicitly, there are 2 simple ways:

v) Analytically: write Jac = V/L, V complex vector space, L a lattice. Define:

$$V = \text{dual of } R_1(C)$$

$$L = \left\{ \begin{array}{l} \text{set of } \ell \in V \text{ obtained as periods, i.e.,} \\ \ell(\omega) = \int_\gamma \omega \quad \text{for some 1-cycle } \gamma \text{ on } C. \end{array} \right.$$

Fixing a base point $a_o \in C$, define for all $a \in C$

$$\phi(a) = \left\{ \begin{array}{l} \text{image in } V/L \text{ of any } \ell \in V \text{ defined by} \\ \qquad \ell(\omega) = \int_{a_o}^{a} \omega, \\ \text{where we fix a path from } a_o \text{ to } a. \end{array} \right.$$

Note that since Jac is a group,

$$V^* \cong \left(\begin{array}{c} \text{translation-invariant} \\ \text{1-forms on Jac} \end{array} \right) \cong \left(\begin{array}{c} \text{cotangent sp. to Jac at } \alpha \\ \text{any } \alpha \in \text{Jac} \end{array} \right) \cong R_1(C).$$

vi) Algebraically: following Weil's original idea, introduce $S^gC = C \times \cdots \times C/S_g$ and construct by the Riemann-Roch theorem, a "group-chunk" structure on S^gC, i.e., a partial group law:

$$m: \ U_1 \times U_2 \longrightarrow U_3$$

$$U_i \subset S^gC \quad \text{Zariski-open.}$$

He then showed that any such algebraic group-chunk prolonged automatically into an algebraic group J with $S^gC \supset U_4 \subset J$ (some Zariski-open U_4).

An important point is that ϕ is an integrated form of the canonical map $\Phi: C \longrightarrow \mathbb{P}^{g-1}$ discussed at length above –

vii) Φ is the Gauss map of ϕ, i.e., for all $x \in C$, $d\phi(T_{x,C})$ is a 1-dimensional subspace of $T_{\phi(x),\mathrm{Jac}}$, and by translation this is isomorphic to Lie(Jac). If $\mathbb{P}^{g-1}$ = [space of 1-dim$\frac{1}{}$ subsp. of Lie(Jac)], then $d\phi: C \longrightarrow \mathbb{P}^{g-1}$ is just Φ.

(Proof: this is really just a rephrasing of (iv).)

The Jacobian has always been the corner-stone in the analysis of algebraic curves and compact Riemann surfaces. Its power lies in the fact that it <u>abelianizes</u> the curve and is a <u>reification</u> of H_1, e.g.,

viii) Via ϕ: $C \longrightarrow \mathrm{Jac}$, every abelian covering $\pi: C_1 \longrightarrow C$ is the "pull-back" of a unique covering $p: G_1 \longrightarrow \mathrm{Jac}$ (i.e., $C_1 \cong C \times_{\mathrm{Jac}} G_1$).

Weil's construction in vi) above was the basis of his epoch-making proof of the Riemann Hypothesis for curves over finite fields, which really put characteristic p algebraic geometry on its feet.

There are very close connections between the geometry of the curve C (e.g., whether or not C is hyperelliptic) and Jac. We want to describe these next in order to tie in Jac with the special cases studied in Lecture I, and in order to "see" Jac very concretely in low genus. The main tool we want to use is: