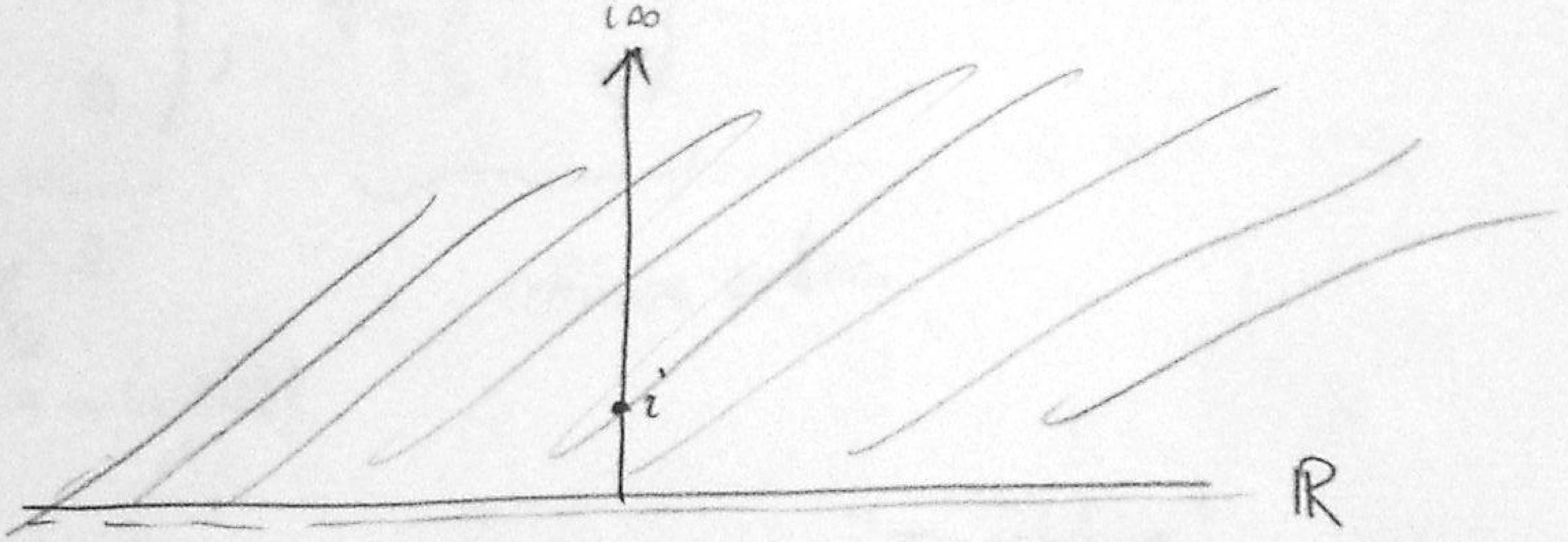
$SL_2(\mathbb{Z})$

Follow Serre Ch VII, § 1.1–1.2 very closely.

$\mathfrak{h} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ = complex upper half plane



$SL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{R} \text{ and } ad-bc=1 \right\}$

linear fractional transformations

$\mathbb{C} \cup \{\infty\}$ $\qquad z \longmapsto \dfrac{az+b}{cz+d}$

<u>Exercise</u>: $\mathrm{Im}(gz) = \dfrac{\mathrm{Im}(z)}{|cz+d|^2}$ $\qquad$ [use that $\mathrm{Im}(z) = \frac{1}{2}(z-\bar{z})$

(box this on side board)

$\Downarrow$

$\mathfrak{h} \circlearrowleft SL_2(\mathbb{R})$.

$\circlearrowleft$ $PSL_2(\mathbb{R}) = SL_2(\mathbb{R}) / \{\pm 1\}$ also acts on $\mathfrak{h}$

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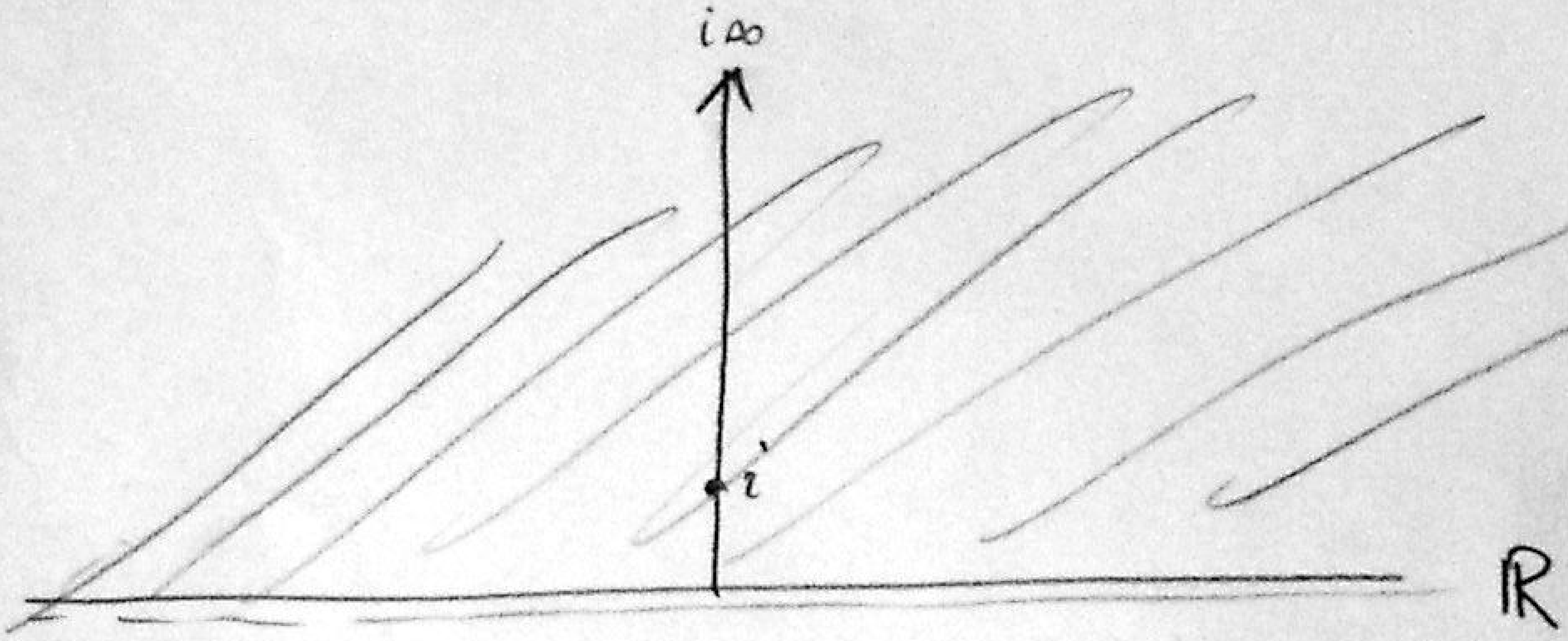

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The Modular Group

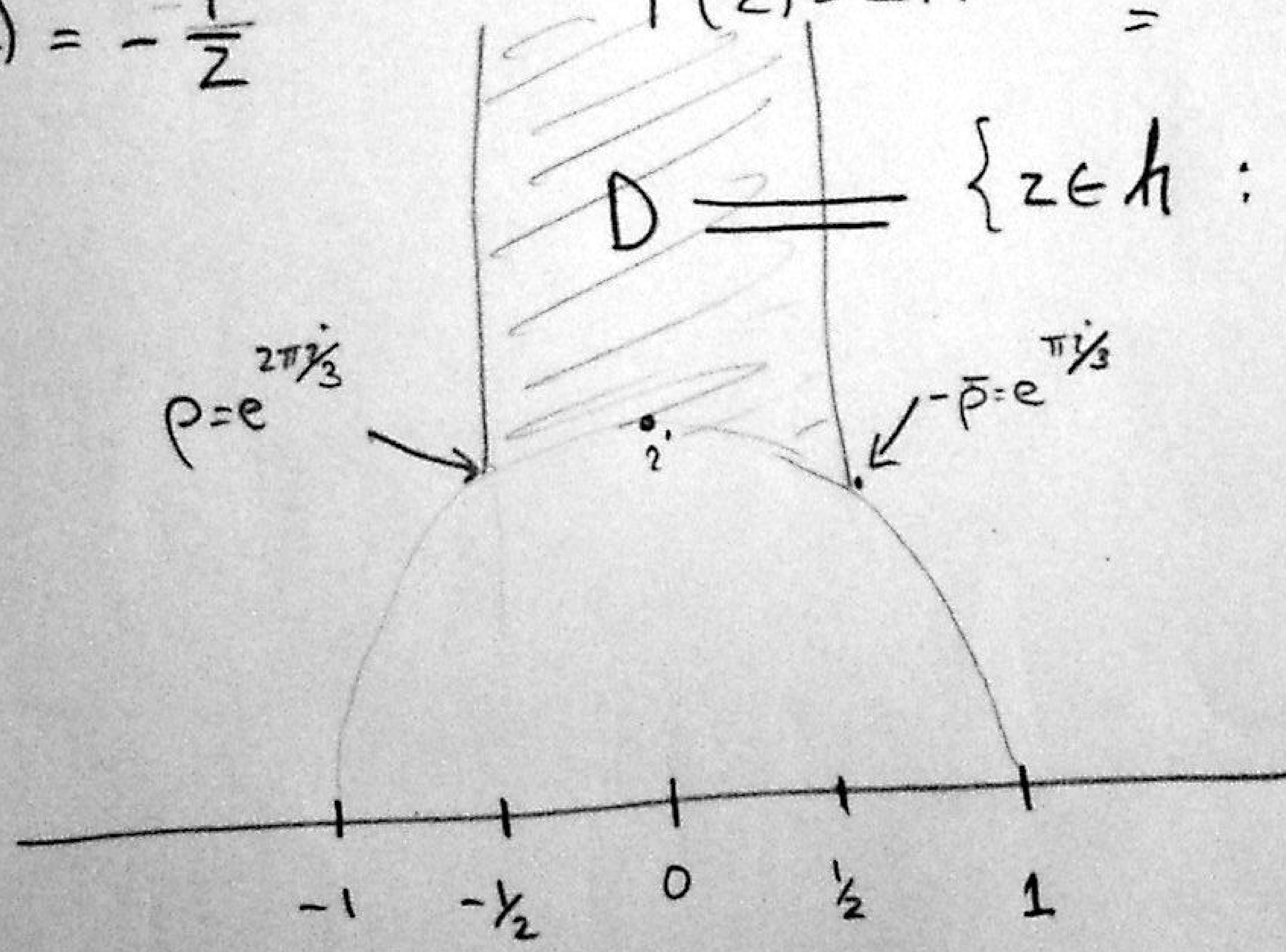
$$G := PSL_2(\mathbb{Z}) := SL_2(\mathbb{Z}) / \{\pm 1\}.$$

Let

$$S = \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{\substack{\text{order 2} \\ \text{in } G \\ (\text{order 4 in } SL_2(\mathbb{Z}))}}, \quad T = \underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}_{\text{infinite order,}}$$

$S(z) = -\frac{1}{z}$ $\qquad$ $T(z) = z+1$ "Translation"

$D = \{ z \in \mathfrak{h} : |z| \geq 1, \ |\mathrm{Re}(z)| \leq \frac{1}{2} \}$



(play with fundomain Java program)

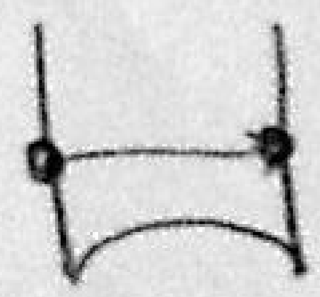
Theorem:

A. D is a fundamental domain for action of G on $\mathfrak{h}$ in the following sense:

(1) If $z \in \mathfrak{h}$, $\exists\, g \in G$ s.t. $g(z) \in D$.] move everything into D

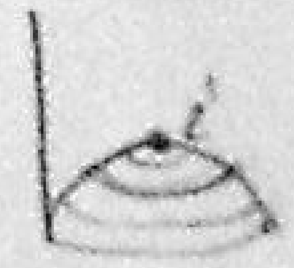
(2) If $z \neq z' \in \mathfrak{h}$ and $\exists\, g \in G$ s.t. $gz = z'$ then

$$\mathrm{Re}(z) = \pm \frac{1}{2} \text{ and } z = z' \pm 1$$



or

$$|z| = 1 \text{ and } z' = -\frac{1}{z}$$



(3) $z \in D$ and $I(z) =$ Stabilizer of z.

$\Longrightarrow I(z) = \{1\}$, except if

$z = i \quad (I(z) = \{1, S\})$

$z = \rho \quad (I(z) = \{1, ST, (ST)^2\})$

$z = -\bar{\rho} \quad (I(z) = \{1, TS, (TS)^2\})$

B: G is generated by S and T.

Proof (Rest of class)

Let $G' := \langle S, T \rangle \leq G$.

<u>Proof of A1:</u> Suppose $z \in h$.
(say in words)

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G' \Rightarrow \mathrm{Im}(gz) = \frac{\mathrm{Im}(z)}{|cz+d|^2}$$

$\#\{(c,d) \in \mathbb{Z} \times \mathbb{Z} : |cz+d| \leq B\}$ is finite for any B.

$$\Downarrow$$

there is $g \in G'$ such that $|\mathrm{Im}(gz)|$ is maximized, [note that $\mathrm{Im}(\,) > 0$ since we're in upper halfplane.]

Choose n s.t. $|\mathrm{Re}(T^n g z)| \leq \frac{1}{2}$.

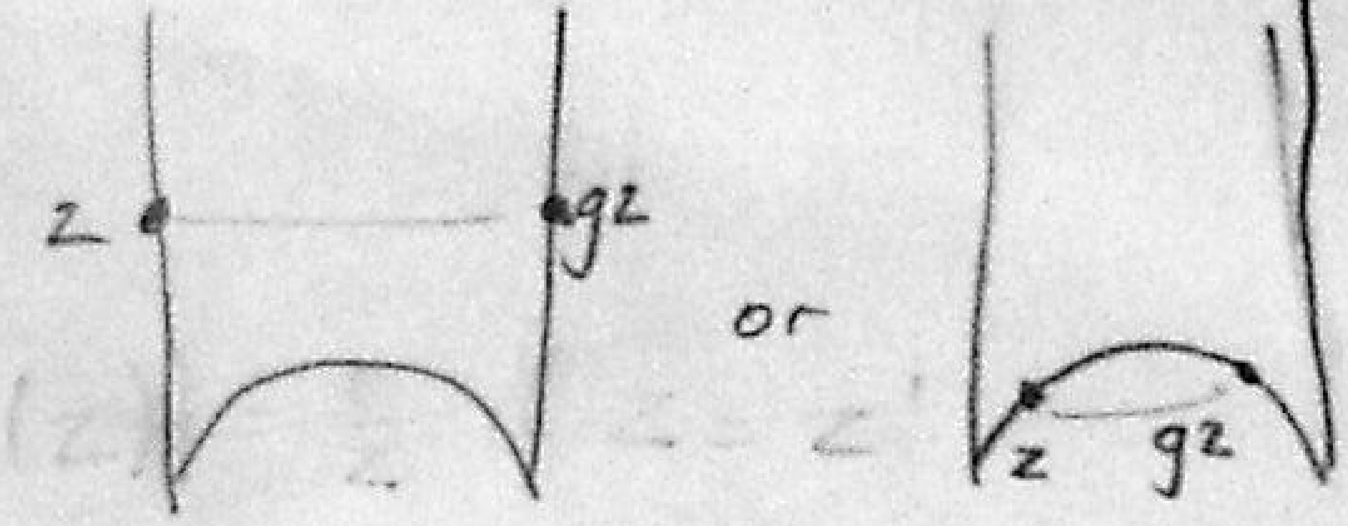
<u>Claim:</u> $\boxed{z' = T^n g z \in D.}$

If $|z'| < 1$ then $|\mathrm{Im}(-\frac{1}{z'})| > |\mathrm{Im}(z')|$, by Exercise! $\mathrm{Im}(z')/|z|^2$

which contradicts maximality of $\mathrm{Im}(z')$.

Proof of A2 & 3:

Suppose $z, gz \in D$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\xRightarrow{\text{want}}$



and $i, \rho, -\bar{\rho}$ only fixed points

Assume $\mathrm{Im}(gz) \geq \mathrm{Im}(z)$

(otherwise replace z, g by gz, g^{-1})

$\Downarrow$

$$|cz+d| = \sqrt{\frac{\mathrm{Im}(z)}{\mathrm{Im}(gz)}} \leq 1$$

$\Downarrow$ (since $|z| \geq 1$ and adding d goes sideways)

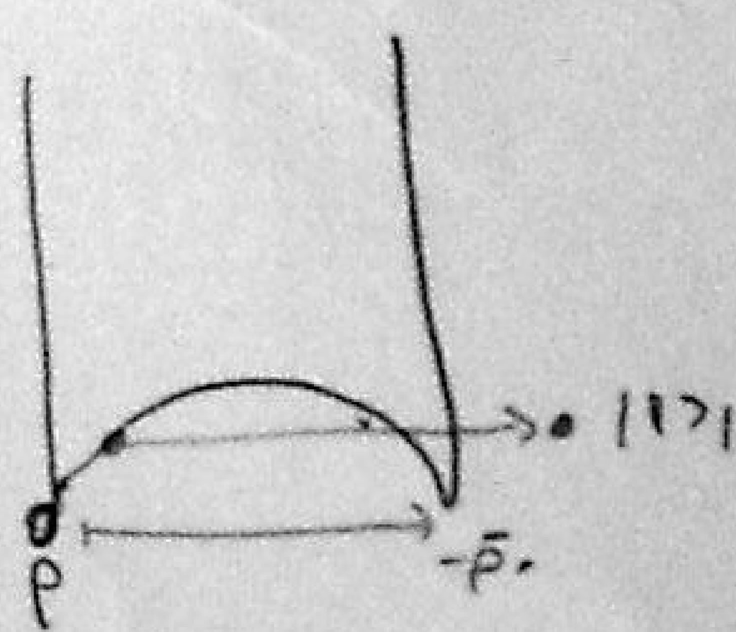
$|c| \leq 1$

Three cases: $c = -1, 0, 1$:

<u>$c=0$:</u> $\Rightarrow d = \pm 1 \Rightarrow g =$ translation by $\pm b$

$\Rightarrow$ what we want since $|\mathrm{Re}(z)| \leq \frac{1}{2}$ and $|\mathrm{Re}(gz)| \leq \frac{1}{2}$.

<u>$c=1$:</u> $|z+d| \leq 1 \Rightarrow d = 0$ except when $z = \rho$ or $-\bar{\rho}$ when d can be 0 or 1 (resp. $0, -1$)



$\Downarrow$

$|z| \leq 1$ ✓

$\Downarrow$

$|z| = 1.$

$$gz = \frac{az-1}{z} = a - \frac{1}{z}$$

since $ad - bc = 1$

(reflect through y-axis and add a)

$\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$

$\Downarrow$

$a = 0$ <u>except</u> if $z = \rho$ or $-\bar{\rho}$

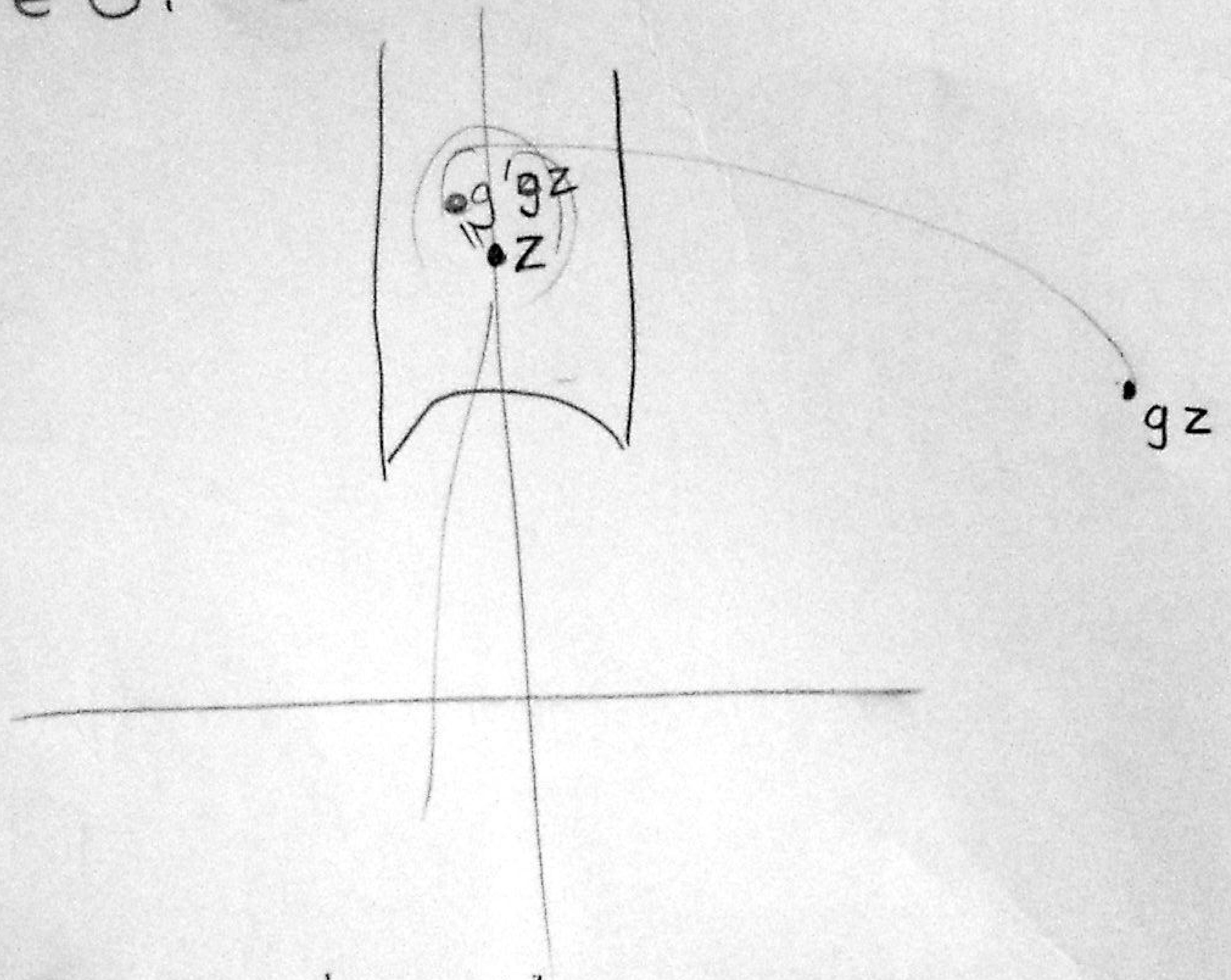
$a = -1$ $\quad$ $a = 1$

<u>$c=-1$:</u> replace g by $-g$, which changes nothing.

(exact stabilizers omitted)

Proof of B: I.e. $G' = G$.

Let $g \in G$.



$g'gz = z$ since they are conjugate by G.

Also Stabilizer$(z) = \{1\}$.

$$\Downarrow$$

$g'g = 1 \Rightarrow g = (g')^{-1} \in G'$,

So $G = G'$.

$\boxed{\times}$

Next Time:

$$
\begin{tikzcd}
 & SL_2(\mathbb{Z}) \arrow[r, two heads] & SL_2(\mathbb{Z}/N\mathbb{Z}) \\
\Gamma_0(N) \arrow[ur, dash] \arrow[rr] & & {} \\
 & \Gamma_1(N) \arrow[ul, dash] \arrow[r] & {} \\
 & \Gamma(N) \arrow[u, dash] \arrow[r] & \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}_{\bmod N}
\end{tikzcd}
$$

generators?

How to deal with these congruence subgroups.

Also how to get Riemann surfaces from them.