Homework Assignment 7

(Math 252: Modular Abelian Varieties)

William A. Stein

Oct. 29 (Due: Nov. 5))

1. Let L be a lattice in $\mathbf{C}$ and let n be a positive integer. Prove that the number of sublattices of L of index n is equal to the sum of the positive divisors of n. (Hint: Use the characterization of sublattices of index n from the notes, and reduce to the prime power case.)

2. In this problem you probably want to use a computer, though it isn't strictly necessary.

   (a) Show that $k = 24$ is the smallest weight such that $\dim S_k(1) > 1$.

   (b) Find a basis for $S_{24}(1)$.

   (c) Compute a matrix for the Hecke operator T_2 with respect to your basis.

   (d) Compute the characteristic polynomial of your matrix.

   (e) Prove that the characteristic polynomial of your matrix is irreducible, and hence verify Maeda's conjecture in this case.

   (f) Find the smallest k such that $S_k(1)$ has dimension 3.

   (g) Compute a matrix for T_2 on $S_k(1)$.

   (h) Verify that the characteristic polynomial of T_2 has Galois group the full symmetric group S_3, which verifies one case of what Buzzard proves in his paper on Maeda's conjecture.

3. Let m and d be positive integers. Prove that

$$\sum_{b=0}^{d-1} \left(e^{(2\pi i m)/d}\right)^b$$

   is nonzero if and only if $d \mid m$.

4. Let V be a finite-dimensional complex vector space equipped with an action of $(\mathbf{Z}/N\mathbf{Z})^*$. For each homomorphism $\chi : (\mathbf{Z}/N\mathbf{Z})^* \to \mathbf{C}^*$, let

$$V(\chi) = \{v \in V : av = \chi(a)v \text{ all } a \in (\mathbf{Z}/N\mathbf{Z})^*\}.$$

   Prove that $V = \oplus_\chi V(\chi)$, where the sum is over all $\chi : (\mathbf{Z}/N\mathbf{Z})^* \to \mathbf{C}^*$.