Homework Assignment 1
Math 252: Modular Abelian Varieties

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Sep. 17 (Due: Sep. 24)

There are 7 problems, each worth an equal number of points.

1. Verify that if $g \in \mathrm{SL}_2(\mathbf{R})$ and $z \in \mathbf{C}$ then
$$\mathrm{Im}(g(z)) = \frac{\mathrm{Im}(z)}{|cz+d|^2}.$$

2. Let D be the subset of points $z \in \mathbf{C}$ with $|z| \geq 1$ and $|\mathrm{Re}(z)| \leq 1/2$. For each z in the following list, find an element $g \in \mathrm{SL}_2(\mathbf{Z})$ such that $g(z) \in D$:
$$-1 + \frac{i}{2}, \quad -\frac{1}{5} + 3i, \quad \frac{7}{8} + \frac{4}{5}i.$$

3. (a) Find an element $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$ such that $c = 7$ and $d = 9$.
   (b) Write the element you found in (a) in terms of $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

4. (a) Find an element of $g \in \mathrm{SL}_2(\mathbf{Z})$ whose reduction modulo 5 is
$$\begin{pmatrix} 1 & 1 \\ 6 & 12 \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}/5\mathbf{Z}).$$
   (b) Let N be a positive integer. Prove that the reduction mod N map from $\mathrm{SL}_2(\mathbf{Z})$ to $\mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})$ is surjective. [Hint: If you get stuck, you can find the proof somewhere in Shimura's *Introduction to the Arithmetic Theory of Automorphic Functions* and reformulate it in your own way.]

5. Let N be a positive integer.Compute the index of each of the following groups in $\mathrm{SL}_2(\mathbf{Z})$:
$$\Gamma(N), \quad \Gamma_1(N), \quad \Gamma_0(N)$$

6. Compute generators for $\Gamma_0(3)$, $\Gamma_1(3)$, and $\Gamma(3)$.

7. Let p be a prime. Prove that $\Gamma_0(p)\backslash\mathbf{P}^1(\mathbf{Q})$ has cardinality 2. What about $\Gamma_0(p^2)\backslash\mathbf{P}^1(\mathbf{Q})$?