Homework Assignment 5
Due Wednesday October 30

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Math 124 HARVARD UNIVERSITY **Fall 2002**

Instructions: *Please work with others.* There are **six** problems.

1. (4 points) Construct a sequence of positive real numbers $a_0, a_1, a_2, \ldots$ such that
$$\lim_{n\to\infty} c_{2n} \neq \lim_{n\to\infty} c_{2n+1},$$
where $c_n = [a_0, \ldots, a_n]$ is the nth partial convergent. To get full credit on this problem, you do not have to compute the limits for your example, though it would be cool if you were to do so.

2. (2 points) Find two distinct continued fractions $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ such that
$$[a_0, a_1, a_2, \ldots] = [b_0, b_1, b_2, \ldots].$$
(Note that necessarily the a_i and b_i won't all be integers.)

3. (a) (3 points) Find the continued fraction expansion of $(1+2\sqrt{3})/4$. Prove that your answer is correct.
   (b) (3 points) Evaluate the infinite continued fraction $[0, \overline{1, 3}]$

4. (5 points) Let $a_0 \in \mathbb{R}$ and $a_1, \ldots, a_n$ and b be positive real numbers. Prove that
$$[a_0, a_1, \ldots, a_n + b] < [a_0, a_1, \ldots, a_n]$$
if and only if n is odd.

5. Let $s(n) = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ be the sum of the first n positive integers.
   (a) (2 points) Use MAGMA to find all $n < 10000$ such that $s(n)$ is a perfect square.
   (b) (5 points) Prove that there are infinitely many n such that $s(n)$ is a perfect square. (Hint: Find a relationship between such n and solutions to a certain Pell's equation.)

6. Let α be a real number, and let p_k/q_k denote the partial convergents of the integral continued fraction for α.
   (a) (2 points) Prove that for every $k \geq 0$,
$$\left|\alpha - \frac{p_k}{q_k}\right| < q_k^2.$$
   (b) (5 points) Let the decimal expansion of α be
$$\alpha = b + \frac{b_1}{10} + \frac{b_2}{10^2} + \frac{b_3}{10^3} + \frac{b_4}{10^4} + \cdots,$$
where $0 \leq b_n \leq 9$ for all n. Suppose that for some convergent p_k/q_k we have $q_k = 100$. Prove that either $b_3 = b_4 = 0$ or $b_3 = b_4 = 9$. (This problem is from page 210 of Stark's book *An Introduction To Number Theory*.)