Math 581g, Fall 2011, Homework 2

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Due: Friday, October 14, 2011

There are 7 problems. Turn your solutions in Friday, October 14, 2011 in class. You may work with other people and can find the LaTeX of this file at http://wstein.org/edu/2011/581g/hw/. Ask me questions during my office hours 11:00am-3:15pm on Thursday, October 13 in Sieg 311.

1. (Easy warm up) Suppose $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ is a lattice in $\mathbb{C}$. Prove that either ω_1/ω_2 or ω_2/ω_1 is in the complex upper half plane.

2. (Warm up) Let M_k denote the space of modular forms of weight k and level 1. Prove that if $k \geq 2$ and $f \in M_k$ is a constant function, then $f = 0$.

3. Let E be an elliptic curve over $\mathbb{C}$ given by a Weierstrass equation $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$. Prove that the differential $\omega = \frac{dx}{2y+a_1x+a_3}$ has no poles. You may follow the proof presented in class in the special case when $a_1 = a_2 = a_3 = 0$. [Though you can read a complete proof of this in Silverman's book on elliptic curves, I encourage you not to.]

4. Let K be a number field and ℓ a prime number. Prove that
$$K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \cong \prod_{\lambda \mid \ell} K_\lambda.$$
Here $\lambda \mid \ell$ are the prime ideals of the ring of integers of K that contain ℓ and K_λ is the completion of K at λ.

5. Let E be the elliptic curve $y^2 = x(x-1)(x+1)$. Show that the representation $\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F}_2)$ that gives the action of the Galois group on $E[2]$ is reducible, i.e., has an invariant subspace of dimension 1.

6. In the section of the textbook called *Modular forms as functions on lattices* we define maps between the set $\mathcal{R}$ of lattices in $\mathbb{C}$ and the set $\mathcal{E}$ of isomorphism classes of pairs (E, ω), where E is an elliptic curve over $\mathbb{C}$ and $\omega \in \Omega^1_E$ is a nonzero holomorphic differential 1-form on E. Prove that the maps in each direction defined in the book are bijections.

7. Prove that the number of subgroups of $\mathbb{Z}^2$ of index n is equal to the sum of the positive divisors of n. [Hint: first do the case $n = p$ is prime first as a warm up, then reduce to the prime power case.]