Lecture 13: Brauer Groups

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This lecture is about Brauer groups.
Reference: Chapter X of Serre's *Local Fields*.

1 The Definition

Let k be a field, and fix a separable closure $k^{\rm sep}$ of k.

Definition 1.1. The *Brauer group* of k is

$$\mathrm{Br}_k = \mathrm{H}^2(k, (k^{\rm sep})^*).$$

The Brauer group of a field is a measure of the complexity of the field. It also plays a central role in duality theorems, and in class field theory.

1.1 Some Motivating Examples

1. Let E be an elliptic curve over k and n a positive integer coprime to $\mathrm{char}(k)$. Consider the Weil pairing
$$E[n] \otimes E[n] \to \mu_n.$$
Cup product defines a map
$$\mathrm{H}^1(k, E[n]) \otimes \mathrm{H}^1(k, E[n]) \to \mathrm{H}^2(k, \mu_n).$$
The inclusion $\mu_n \hookrightarrow (k^{\rm sep})^*$ defines a homomorphism
$$\mathrm{H}^2(k, \mu_n) \to \mathrm{H}^2(k, (k^{\rm sep})^*) = \mathrm{Br}_k\,.$$
We thus have a pairing on $\mathrm{H}^1(k, E[n])$ with values Br_k. It would thus be very handy to understand Brauer groups better.

2. If A is a simple abelian variety over k, then $R = \mathrm{End}(A) \otimes k$ is a division algebra over k. Its center is an extension F of k, and R is a central simple F-algebra. As we will see later, the isomorphism classes of central simple F-algebras are in natural bijection with the elements of Br_F. It would thus be very handy, indeed, to understand Brauer groups better.

2 Examples

Recall that if G is a finite *cyclic* group and A is a G-module, then $\hat{\mathrm{H}}^{2q}(G, A) \approx \hat{\mathrm{H}}^0(G, A)$ and $\hat{\mathrm{H}}^{2q+1}(G, A) \approx \hat{\mathrm{H}}^1(G, A)$, a fact we proved by explicitly writing down the following very simple complete resolution of G:

$$\cdots \to \mathbb{Z}[G] \xrightarrow{s-1} \mathbb{Z}[G] \xrightarrow{N} \mathbb{Z}[G] \xrightarrow{s-1} \mathbb{Z}[G] \xrightarrow{N} \mathbb{Z}[G] \to \cdots,$$

where $N = \sum s^i$ is the norm.

Proposition 2.1. *The Brauer group of the field* $\mathbb{R}$ *of real numbers has order* 2.

Proof. We have $\mathbb{C} = \mathbb{R}^{\mathrm{sep}}$, and $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ is cyclic of order 2. Thus

$$\mathrm{Br}_{\mathbb{R}} = \mathrm{H}^2(G, \mathbb{C}^*) \cong \hat{\mathrm{H}}^0(G, \mathbb{C}^*) \approx (\mathbb{C}^*)^G / N\mathbb{C}^* \cong \mathbb{R}^*/\mathbb{R}^*_+ \cong \{\pm 1\}.$$

□

Lemma 2.2. *Suppose G is a finite cyclic group and A is a finite G-module. Then*

$$\#\hat{\mathrm{H}}^q(G, A) = \#\hat{\mathrm{H}}^0(G, A)$$

for all $q \in \mathbb{Z}$, i.e., $\#\hat{\mathrm{H}}^q(G, A)$ is independent of q.

Proof. Since, as was mentioned above, $\hat{\mathrm{H}}^{2q}(G, A) \approx \hat{\mathrm{H}}^0(G, A)$ and $\hat{\mathrm{H}}^{2q+1}(G, A) \approx \hat{\mathrm{H}}^1(G, A)$, it suffices to show that $\#\hat{\mathrm{H}}^{-1}(G, A) = \#\hat{\mathrm{H}}^0(G, A)$. Let s be a generator of G. We have an exact sequence

$$0 \to A^G \to A \xrightarrow{s-1} A \to A/(s-1)A \to 0.$$

Since every term in the sequence is finite,

$$\#A^G = \#(A/(s-1)A).$$

Letting $N_G = \sum s^i$ be the norm, we have by definition an exact sequence

$$0 \to \hat{\mathrm{H}}^{-1}(G, A) \to A/(s-1)A \xrightarrow{N_G} A^G \to \hat{\mathrm{H}}^0(G, A) \to 0.$$

The middle two terms in the above sequence have the same cardinality, so the outer two terms do as well, which proves the lemma. □

Proposition 2.3. *If k is a finite field, then* $\mathrm{Br}_k = 0$.

Proof. By definition,

$$\mathrm{Br}_k = \mathrm{H}^2(k, \overline{k}^*) = \varinjlim_F \mathrm{H}^2(F/k, F^*),$$

where F runs over finite extensions of k. Because $G = \mathrm{Gal}(F/k)$ is a finite cyclic group, Lemma 2.2 and triviality of the first cohomology of the multiplicative group of a field together imply that

$$\#\,\mathrm{H}^2(F/k, F^*) = \#\hat{\mathrm{H}}^1(F/k, F^*) = 1.$$

□

Example 2.4. The following field all have $\mathrm{Br}_k = 0$.

1. Let k be any algebraically or separably closed field. Then $\mathrm{Br}_k = 0$, obviously, since $k^{\mathrm{sep}} = k$.

2. Let k be any extension of transcendance degree 1 of an algebraically closed field. Then $\mathrm{Br}_k = 0$. (See §X.7 of Serre's *Local Fields* for references.)

3. Let k be the maximal unramified extension K^{ur} of a local field K with perfect residue field (e.g., the maximal unramified extension of a finite extension of $\mathbb{Q}_p$). Then $\mathrm{Br}_k = 0$. (See §X.7 of Serre's *Local Fields* for references.)

4. Let k be any algebraic extension k of $\mathbb{Q}$ that contains *all* roots of unity (thus k is necessarily an infinite degree extension of $\mathbb{Q}$). Then $\mathrm{Br}_k = 0$.

The following theorem is one of the main results of *local class field theory.*

Theorem 2.5. *Let k be a local field with perfect residue field (e.g., a finite extension of $\mathbb{Q}_p$). Then $\mathrm{Br}_k \cong \mathbb{Q}/\mathbb{Z}$.*

The following theorem is one of the main results of *global class field theory.*

Theorem 2.6. *Let k be a number field, and for any place v of k, let k_v be the completion of k at v, so k_v is a p-adic local field, $\mathbb{R}$, or $\mathbb{C}$. We have a natural exact sequence*

$$0 \to \mathrm{Br}_k \to \bigoplus_v \mathrm{Br}_{k_v} \xrightarrow{(x_v)\mapsto\sum x_v} \mathbb{Q}/\mathbb{Z} \to 0,$$

We obtain the map to $\mathbb{Q}/\mathbb{Z}$ by using Theorem 2.5 to view each Br_{k_v} as $\mathbb{Q}/\mathbb{Z}$, and we view $\mathrm{Br}_{\mathbb{R}} = \frac{1}{2}\mathbb{Z}/\mathbb{Z}$.

3 Brauer Groups and Central Simple Algebras

Definition 3.1. Let k be a field. Then a *central simple k-algebra* is a finite dimensional k-algebra A that satisfies any one of the following equivalent conditions:

1. A has no nontrivial two-sided ideals, and A has center k.

2. The algebra $A_{\overline{k}} = A \otimes_k \overline{k}$ is isomorphic to a matrix algebra over $\overline{k}$.

3. There is a finite extension F/k such that A_F is isomorphic to a matrix algebra over F.

4. A is isomorphic to a matrix algebra over a division algebra D with center k.

We say that two central simple k-algebras are equivalent if the corresponding division algebras D in 4 above are k-isomorphic. Tensor product endows the set of equivalence classes of central simple k-algebras with the structure of abelian group.

Theorem 3.2. *The group $\mathcal{B}_k$ of equivalence classes of central simple k-algebras is isomorphic to the Brauer group Br_k.*

The proof of Theorem 3.2 is somewhat involved. We will content ourselves with sketching some of the main ideas; in particular, we will explicitly construct the homomorphism $\mathcal{B}_k \to \mathrm{Br}_k$, but will not prove that it is an isomorphism (the argument, which uses descent, is given in Serre's *Local Fields*).

Fix a finite Galois extension F of k and let $\mathcal{B}(n, F/k)$ be the set of equivalence classes of central simple k-algebras A such that $A_F \approx M_n(F)$, where $M_n(F)$ is the algebra of $n \times n$ matrices over F. Then $\mathcal{B}$ is the union of all $\mathcal{B}(n, F/k)$ over all n and F.

Given $A \in \mathcal{B}(n, F/k)$, let $\varphi : A_F \to M_n(F)$ be a fixed choice of isomorphism. Define a set-theoretic map

$$f : \mathrm{Gal}(F/k) \to \mathrm{Aut}_F(M_n(F)) \approx \mathrm{PGL}_n(F)$$

by

$$f(s) = \varphi^{-1} \circ s(\varphi) = \varphi^{-1} \circ s \circ \varphi \circ s^{-1}$$

Then

$$[f] \in \mathrm{H}^1(F/k, \mathrm{PGL}_n(F)),$$

where this H^1 is a cohomology *set* (!).

Proposition 3.3. *The above construction $A \mapsto [f]$ defines a bijection between $\mathcal{B}(n, F/K)$ and $\mathrm{H}^1(F/k, \mathrm{PGL}_n(F))$.*

(The above proposition is proved in Serre's *Local Fields.*)

Consider the exact sequence

$$1 \to F^* \to \mathrm{GL}_n(F) \to \mathrm{PGL}_n(F) \to 1.$$

There is a well defined connecting homorphism

$$\mathrm{H}^1(F/k, \mathrm{PGL}_n(F)) \to \mathrm{H}^2(F/k, F^*).$$

Since $\mathrm{H}^2(F/k, F^*) \xrightarrow{\mathrm{inf}} \mathrm{Br}_k$, we thus obtain a natural map

$$\mathcal{B}(n, F/K) \to \mathrm{Br}_k\,.$$

This induces the claimed isomorphism $\mathcal{B} \to \mathrm{Br}_k$.

Next time: Galois cohomology of abelian varieties. Principal homogenous spaces. Hopefully, a complete proof that for any abelian variety over a finite field k, we have $\mathrm{H}^q(k, A) = 0$ for all $q \geq 1$.