Plan: ref: Serre.
· defn of group homo
· basic properties
· defn of $\hat{H}^q$
· basic props.
· cores
· cup prod.
· finite cyclic
· Galois cohomology

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Lecture 8: Group Homology; Tate Cohomology $\hat{H}^q$

§1. Homology: impossible to avoid ü.

G - group

A - G-module (abelian!)

$DA = \langle s.a - a : a \in A, s \in G \rangle$ subgroup generated by.

Observation: DA - G-module $\quad tst^{-1}ta$

$$t.(s.a - a) = t.s.a - t.a$$
$$= (tst^{-1}).(ta) - ta \in DA.$$

So A/DA is G-module. with trivial G-action.

$H_0(G,A) = A/DA.$ = largest quotient of A with trivial G-action.

$H_q(G,A) = \mathrm{Tor}_q^{\mathbb{Z}[G]}(\mathbb{Z}, A)$

$H_q(G,-)$ = left-derived functors of $H_0(G,-)$.

Long exact sequence:

$$0 \to A \to B \to C \to 0$$

$$\cdots \to H_1(G,B) \to H_1(G,C) \xrightarrow{\delta} H_0(G,A) \to H_0(G,B) \to H_0(G,C) \to 0.$$

Interesting examples: $H_1(G,\mathbb{Z}) = G/G' = G^{ab}$, G' = commutator subgroup $= \langle aba^{-1}b^{-1} : a,b \in G \rangle$

(See Serre §VII.4 for proof.)

§2. Tate Cohomology Groups:

- Naturally relate homology and cohomology
- Make several results much more natural.

G - group, finite. A, G-module

$$N = \sum_{s \in G} s \quad \text{"norm"}$$

$\boxed{\text{links homology and cohomology!}}$

$$N: A \longrightarrow A$$
$$a \longmapsto N(a) = \sum_{s \in G} sa$$

G-module homomorphism, since $t \sum sa = \sum tsa = \sum s a$ (permutes!)

Augmentation ideal:

$$I_G = (s-1 : s \in G) \subseteq \mathbb{Z}[G]$$

Lemma. (1) $I_G A \subseteq \ker(N) \stackrel{\text{def}}{=} A[N]$ and (2) $\mathrm{Im}(N) \subseteq A^G$.

($I_G A$ = DA from above.)

Proof: (1) $N((s-1)a) = \left(\sum_{t \in G} t\right) \cdot (s-1)a = \left(\sum_{t \in G} ts - \sum_{t \in G} t\right) a = 0$ (permute.)

(2) $s N(a) = s \sum_{t \in G} t a = \sum st a = \sum t a = N(a)$. (permute) $\blacksquare$

Get induced map:

$$\begin{array}{ccc} A & \xrightarrow{N} & A \\ \downarrow & & \cup \\ N^*: H_0(G,A) & \longrightarrow & H^0(G,A) \\ \| & & \| \\ A/I_G A & & A^G \end{array}$$

$$
\begin{array}{ccc}
H_0(G,A) & \xrightarrow{N^*} & H^0(G,A) \\
\cup| & & \Downarrow \\
\hat{H}_0(G,A) = \mathrm{Ker}(N^*) & & \hat{H}^0(G,A) = \mathrm{coker}(N^*) \\
\| & & \| \\
A[N]/I_G A & \xrightarrow{0} & A^G/N(A)
\end{array}
$$

<u>Defn</u>:

<u>Prop</u>: If A is induced then $\hat{H}^0(G,A) = 0$.

<u>Proof</u>: Induced <u>means</u> $A \cong \bigoplus_{s \in G} s.X$ where $X \leq A$ subgroup.
(<u>easy</u> since G finite) $(\cong \mathbb{Z}[G] \otimes_{\mathbb{Z}} X)$.

Thus each $a \in A$ expressed uniquely as

$$a = \sum_{s \in G} s.x_s, \qquad x_s \in X$$

$a \in A^G \Leftrightarrow x_s$ are all equal. $\Leftrightarrow a = N(x_s)$.

So $A^G = N(A)$, as claimed. ∎

(Also, $\hat{H}_0(G,A) = 0$ when A induced $\leftarrow$ will be on homework.

<u>Defn</u> (Tate Cohomology):

$$\hat{H}^q(G,A) = H^q(G,A) \quad \text{if } q \geq 1$$
$$\hat{H}^0(G,A) = A^G/N(A)$$
$$\hat{H}^{-1}(G,A) = A[N]/I_G A$$
$$\hat{H}^{-q}(G,A) = H_{q-1}(G,A) \quad \text{for } q \geq 2.$$

$\hat{H}^0(G,A)$ is <u>not</u> left exact:

Given short exact $0 \to A \to B \to C \to 0$,

<u>Prop</u>: We have a really long exact sequence

$$\cdots \to \hat{H}^{-2}(G,C) \to \boxed{\hat{H}^{-1}(G,A) \to \hat{H}^{-1}(G,B) \to \hat{H}^{-1}(G,C) \xrightarrow{\delta} \hat{H}^{0}(G,A) \to \hat{H}^{0}(G,B) \to \hat{H}^{0}(G,C)} \to \hat{H}^{1}(G,A) \to \hat{H}^{1}(G,B) \to \hat{H}^{1}(G,C) \to \hat{H}^{2}(G,A) \to \cdots$$

<u>Proof</u>: We have exact sequences:

<u>Snake lemma</u>

$$\begin{tikzcd}
\hat{H}^{-2}(G,C) \arrow[d, phantom, "\cong"] & \hat{H}^{-1}(G,A) \arrow[r] \arrow[d, hook] & \hat{H}^{-1}(G,B) \arrow[r] \arrow[d, hook] & \hat{H}^{-1}(G,C) \arrow[d, hook] \arrow[ddr, dashed, bend left] & \\
H_1(G,C) \arrow[r] \arrow[d] \arrow[ur, dashed] & H_0(G,A) \arrow[r] \arrow[d, "N_A^*"] & H_0(G,B) \arrow[r] \arrow[d, "N_B^*"] & H_0(G,C) \arrow[r] \arrow[d, "N_C^*"] & 0 \arrow[d] \\
0 \arrow[r] & H^0(G,A) \arrow[r] \arrow[d, two heads] & H^0(G,B) \arrow[r] \arrow[d, two heads] & H^0(G,C) \arrow[r] \arrow[d, two heads] & H^1(G,A) \\
& \hat{H}^0(G,A) \arrow[r] & \hat{H}^0(G,B) \arrow[r] & \hat{H}^0(G,C) \arrow[ur, dashed] &
\end{tikzcd}$$

$\square$

<u>Fact</u>:
- Every G-module A embeds in an A^* such that
$$\hat{H}^q(G,A^*) = 0$$
for <u>all</u> q.
- Every A is a quotient of some A_* with
$$\hat{H}^q(G,A_*) = 0$$
for <u>all</u> q.

Super useful. See pg 129 of Serre.

<u>Next/Soon</u>:
$$\hat{H}^p(G,A) \otimes \hat{H}^q(G,B) \longrightarrow \hat{H}^{p+q}(G,A \otimes B). \qquad \underline{\text{cup product}}.$$
$$a,\ b \longmapsto a.b$$