Lecture 2010-01-20:

[Prove Res-Inf sequence is exact.]

Cohomology Sets: (pg 123-126 of Serre's Local Fields)

G group

A group with G-action BUT A not nec. abelian!

Examples:

- ~~$A = G$. for any G.~~ (interesting?) !!
  ~~$G < A$, A any group containing G.~~
- G = Galois group over field k

  A = points of any algebraic group over k, e.g.

  $G = \mathrm{Gal}(\bar{k}/k)$, $A = GL_n(\bar{k})$

  $SL_n(\bar{k})$

Defn: $H^0(G,A) = A^G = \{a \in A : s.a = a \text{ all } s \in G\}$

is a subgroup of A: $\quad s.a = a \ \forall s \Rightarrow s(ab) = ab \ \forall s.$

$H^1(G,A)$: $(1\text{-cocycles})/(\text{equiv relation}) =$ pointed set

1-cocycle: $G \xrightarrow{\text{set-theoretic map}} A$, $s \longmapsto a_s$ s.t. $a_{st} = a_s \cdot s(a_t)$. (write mult. to remind not abelian)

equivalence: $a_s \sim b_s \iff$ there's $a \in A$ with

$$b_s = a^{-1} a_s \cdot s(a) \quad \forall s \in G.$$

$$\left[\begin{array}{l} \text{Why: } s \mapsto s(a) - a \\ b_s = a_s + s(a) - a \end{array} \right]$$

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$H^1(G,A)$ = pointed set.

distinguished element = [1-cocycle]

$a_s = 1$, i.e. all G maps to $1 \in A$.

Functors:

$H^0(G,-)$: NonAb G-modules $\longrightarrow$ Groups.

$H^1(G,-)$: NonAb G-modules $\longrightarrow$ Pointed Sets

E.g. $A \longrightarrow B$ G-homomorphism induces

$$H^0(G,A) \longrightarrow H^0(G,B)$$

$$H^1(G,A) \longrightarrow H^1(G,B).$$

Salvaging the long exact sequence:

$$(*) \quad 1 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 1 \qquad b \longmapsto c$$

exact sequence of Nonabe G-modules.

Want: $C^G \xrightarrow{\delta} H^1(G,A)$

$c \longmapsto [a_s]$

b s.t. $p(b) = c$

$s(b) \equiv b \pmod{i(A)}$ for all $s \in G$ since $p(s(b).b^{-1}) = s(c).c^{-1} = cc^{-1} = 1$ (since $c \in C^G$)

So let $a_s = i^{-1}(b^{-1} s(b)) \in A$.

$s(a \cdot b) = s.a \cdot s.b$

Claim: a_s is well-defined cocycle.

Simplify notation by assuming $A \subset B$. Then

cocycle $\quad a_{st} = b^{-1}(st)(b) = b^{-1}s(b)\,s(b^{-1}t(b)) = a_s \cdot s(a_t)$.

well-defined $\quad$ If $p(b') = p(b) = c$ then $b' = ba$ some $a \in A$ (by exactness).

$$a_s' = a^{-1}b^{-1}s(b)s(a) = a^{-1}a_s\,s(a) \sim a_s.$$

Suppose $A \subseteq$ center of B, so A is abelian, and usual $H^2(G,A)$ defined.

Can define $\Delta : H^1(G,C) \to H^2(G,A)$. (see book)

Proposition:

(1) $1 \to A \xrightarrow{i} B \xrightarrow{p} C \to 1$ any exact sequence nonab. G-modules.

$$1 \to H^0(G,A) \xrightarrow{i_0} H^0(G,B) \xrightarrow{p_0} H^0(G,C) \xrightarrow{\delta} H^1(G,A) \xrightarrow{i_1} H^1(G,B) \xrightarrow{p_1} H^1(G,C)$$

is exact!

(2) if $A \subseteq$ center (B) then

$$H^1(G,B) \xrightarrow{p_1} H^1(G,C) \xrightarrow{\Delta} H^2(G,A)$$

is exact.

See book for proof.