Lecture 2: What is $H^q(G,A)$?

G - arbitrary group

$A \in \mathrm{Mod}_G$ = category of G-modules

Defns: A functor $\mathcal{F}: \mathrm{Mod}_G \to \underset{\text{= abelian groups}}{Ab}$ is exact if

$$0 \to A \to B \to C \to 0 \quad \text{exact}$$

$$\Rightarrow 0 \to \mathcal{F}(A) \to \mathcal{F}(B) \to \mathcal{F}(C) \to 0 \quad \text{is also exact.}$$

or $0 \leftarrow \mathcal{F}(A) \leftarrow \mathcal{F}(B) \leftarrow \mathcal{F}(C) \leftarrow 0$ is exact (if $\mathcal{F}$ is contravariant)

- $A \in \mathrm{Mod}_G$ is projective if

$$B \longmapsto \mathrm{Hom}_G(A,B) \qquad \mathrm{Mod}_G \to Ab.$$

is exact.

- A is injective if $B \mapsto \mathrm{Hom}_G(-,A)$ is exact. (a controvariant functor)

- A is induced if $A \approx \Lambda \otimes_{\mathbb{Z}} X$ some $X \in Ab$, i.e. there's $\underset{\text{subgroup}}{X \subseteq A}$ s.t. $A = \bigoplus_{s \in G} s.X$. (direct sum)

- A is coinduced if $A \approx \mathrm{Hom}_{\mathbb{Z}}(\underset{\mathbb{Z}[G]}{\Lambda}, X)$

Remark: G finite $\Rightarrow$ $\mathrm{Hom}_{\mathbb{Z}}(\Lambda, X) \approx \Lambda \otimes_{\mathbb{Z}} X$. for all X, so induced $\Leftrightarrow$ coinduced.

$$f \longmapsto \sum_{s \in G} s \otimes f(s)$$

Lemma: A is a quotient of some induced module.

Proof: A_0 = group underlying A.

$t.(s\otimes a) = (ts)\otimes a$.

$$\Lambda \otimes A_0 \twoheadrightarrow A$$
$$s\otimes a \longmapsto s.a, \quad \boxed{\text{E}}$$

see homological algebra.

Defn: $H^q(G,-)$ = "right derived functors" of the left exact functor $A \mapsto A^G$.

For $q \geq 0$: $\underset{A^G}{\underset{\parallel}{H^0(G,A)}}, H^1(G,A), H^2(G,A), \ldots$

Long exact sequence:

Functorial in exact sequences: $0 \to A \to B \to C \to 0$ exact

$$0 \to H^0(G,A) \to H^0(G,B) \to H^0(G,C) \xrightarrow{\delta} H^1(G,A) \to H^1(G,B) \to H^1(G,C) \xrightarrow{\delta} \cdots$$

$H^q(G,-)$ is a cohomological δ-functor.

Theorem: $\{H^q(G,-), \delta\}$ is the unique δ-functor such that
(1) $H^0(G,A) = A^G$
(2) A injective $\Rightarrow$ $H^q(G,A) = 0$ for all $q \geq 1$.

Thus everything we can every do/prove must follow from (1) & (2) above!

H^q and Ext:

$$A^G = \mathrm{Hom}_G(\overset{\text{trivial } G \text{ action}}{\mathbb{Z}}, A)$$

$$\boxed{H^q(G,A) = \mathrm{Ext}^q_G(\mathbb{Z},A)}$$

So "compute" $H^q(G,A)$ by:

(1) Resolution of $\mathbb{Z}$ by projective G-modules
$$\cdots \to P_i \to P_{i-1} \to \cdots \to P_0 \to \mathbb{Z} \to 0 \qquad P_i \text{ projective.}$$
exact

(2) Apply $\mathrm{Hom}_G(-,A)$ functor, get $K^i = \mathrm{Hom}_G(P_i, A)$ and complex

$$0 \to \mathrm{Hom}_G(P_0,A) \xrightarrow{d_0^*} \mathrm{Hom}_G(P_1,A) \xrightarrow{d_1^*} \mathrm{Hom}_G(P_2,A) \xrightarrow{d_2^*} \cdots$$

Then $H^q(G,A) = \ker(d_q)/\mathrm{im}(d_{q-1})$.

Next time: Do the above explicitly to get very explicit presentation..

$$P_i = \mathbb{Z}[G^{i+1}] = \mathbb{Z}[\overbrace{G \times \cdots \times G}^{i+1}]$$

G act diagonally.

$$\cdots \xrightarrow{d} P_1 \xrightarrow{d} P_0 \xrightarrow{d} \mathbb{Z} \to 0$$

$$P_i \to P_{i+1} : \quad d(g_0, \ldots, g_i) = \sum_{j=0}^{i} (-1)^j (g_0, \ldots, g_{j-1}, g_{j+1}, \ldots, g_i).$$

etc.

<u>Defn</u>: $A \in \mathrm{Mod}_G$ is <u>relatively injective</u> if it is a direct factor of a coinduced module, i.e.

$$A \oplus B \text{ is coinduced for some } B.$$

<u>Prop</u>: A relatively injective $\Rightarrow H^q(G,A)=0$ for $q\geq 1$.

<u>Proof</u>: Prove for coinduced $\Rightarrow$ general case, since $H^q(G, A\oplus B) = H^q(G,A)\oplus \underbrace{H^q(G,B)}_{\to 0}$

So assume A coinduced, so $A \approx \mathrm{Hom}_{\mathbb{Z}}(\Lambda, X)$, some X.

For $B \in \mathrm{Mod}_G$:
$$\mathrm{Hom}_{\mathbb{Z}}(B,X) \cong \mathrm{Hom}_G(B,A)$$
$$f \longmapsto (b \longmapsto \underbrace{(s \mapsto f(s.b))}_{\in\, \mathrm{Hom}_{\mathbb{Z}}(\Lambda, X)})$$

From above $\mathrm{Hom}_G(P_i, A) = \mathrm{Hom}_{\mathbb{Z}}(P_i, X)$

so $H^q(G,A) = \mathrm{Ext}^q_{\mathbb{Z}}(\mathbb{Z}, X) = 0$ for $q \geq 1$. $\square$

$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, X) \cong X$ so $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, -)$ is exact.

"<u>Dimension shifting</u>":

<u>Cor</u>: $0 \to A \to A^* \to A' \to 0$ exact with A^* co-induced

$$\Longrightarrow H^q(G,A') = H^{q+1}(G,A) \text{ for } q \geq 1.$$

<u>Proof</u>: LES ✓

<u>Note</u>: this shows that the functor $H^{q+1}(G,-)$ is completely determined by $H^i(G,-)$ for $i \leq q$, (giving claimed uniqueness)