

# Schoof-Elkies-Atkin Algorithm for Point Counting on an Elliptic Curve over a Finite Field




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## 1 Introduction


Let  $E$  be an elliptic curve given by the Weierstrass equation


$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$


where the  $a_i$  are integers.  we consider  $E$  as a curve over  $\mathbb{Q}$ , then for any finite field  $\mathbb{F}_q$ , the number of points  $(x, y)$  in  $\mathbb{F}_q \times \mathbb{F}_q$  which satisfy the elliptic curve equation (when taken for all finite fields) characterizes  the isogeny class of the curve  $E$ . If instead we take  $E$  as a curve over some finite field  $\mathbb{F}_q$  from the beginning, then  the number of points of  $E/\mathbb{F}_q$  can help to solve the discrete logarithm problem for two points  $P$  and  $Q$  on  $E$ .

The value  $\#E(\mathbb{F}_q)$ , the cardinality of  $E$  over  $\mathbb{F}_q$ , can be determined by several means. As the most straightforward solution, we could take all points in  $\mathbb{F}_q \times \mathbb{F}_q$ , and see if they satisfy the equation, which would take  $O(q^2)$  operations. We could also express the curve as  $f = y^2 + Ax + B$  (ignoring the case where the characteristic of  $\mathbb{F}_q$  is two), and use the relationship

$$\#E(\mathbb{F}_q) = q + 1 - a_q \text{ and}$$

$$a_q = \sum_{x \in \mathbb{F}_q} \left( \frac{f(x)}{q} \right) \quad \text{$$

where  $\left( \frac{f(x)}{q} \right)$  is the Legendre  symbol. In this way, we would need to compute  $f(x)$  for all  $x \in \mathbb{F}_q$ , which would be  $O(q)$  computations.

Other techniques to compute  $E(\mathbb{F}_q)$  include baby-step / giant-step, which is also exponential time  $O(q^{1/4})$ . 


## 2 Schoof's algorithm

In 1985, René Schoof published a paper describing an algorithm to compute the cardinality of  $E(\mathbb{F}_q)$  for such a curve. If  $q = p^e$  where  $p$  is a large prime, and  $f = y^2 = x^3 + Ax + B$  is the equation for  $E$ , then Hasse's theorem state that  $|a_q| \leq 2\sqrt{q}$ . We can use this short form of the Weierstrass equation for  $E$  because the characteristic of  $\mathbb{F}_q$  is not 2 or 3. Thus, if we can calculate  $a_q$  modulo  $l$  for a set  $S$  of small primes  $l$  such that

$$\prod_{l \in S} l > 4\sqrt{q},$$

then  $a_q$  can be reconstructed using the Chinese Remainder Theorem.

When  $l = 2$  the determination of  $a_q$  modulo  $l$  is straightforward. We know that  $E[2]$  contains  $\mathcal{O}$ , the unique point at infinity, and by Hasse's theorem that  $\#E(\mathbb{F}_q) = q + 1 - a_q$ . Since  $q + 1$  is even, this gives  $\#E(\mathbb{F}_q) \equiv a_q \pmod{2}$ . If  $f = x^3 + Ax + B$  has no root in  $\mathbb{F}_q$ , then  $E(\mathbb{F}_q)$  has no 2-torsion points, and so  $a_q$  is odd. If  $x^3 + Ax + B$  has a root  $(e, 0)$  in  $E(\mathbb{F}_q)$ , then the fact that  $E[2] \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  forces  $\#E(\mathbb{F}_q)$  to be even.

Instead of checking all points in  $\mathbb{F}_q$  to find if  $x^3 + Ax + B$  has a root, we use the fact that the points in  $\mathbb{F}_q$  are exactly  the points satisfying  $x^q - x = 0$ . Thus,  $\gcd(x^q - x, x^3 + Ax + B) = 1 \iff f$  has no root in  $\mathbb{F}_q$ .

## 3 The trace of Frobenius

For  $l > 2$  an odd prime, another method is used to determine  $a_q$  modulo  $l$ . The  $q$ -power Frobenius endomorphism on  $E$

$$\tau : E(\bar{\mathbb{F}}_q) \longrightarrow E(\bar{\mathbb{F}}_q)$$

$$\tau : x \longmapsto x^q$$

$$\tau : \mathcal{O} \longmapsto \mathcal{O}$$

satisfies its characteristic polynomial

$$x^2 - a_q x + q = 0,$$



where  $a_q$  is the trace of the Frobenius element. If  $P = (x, y) \in E(\mathbb{F}_q)$ , this becomes

$$(x^{q^2}, y^{q^2}) + q(x, y) = a_q(x^q, y^q).$$

When  $l$  is an odd prime for which  $\gcd(q, l) = 1$ , and  $E[l]$  is an  $l$ -torsion point of  $E$ , then

$$(x^{q^2}, y^{q^2}) + [q](x, y) \equiv [a_q](x^q, y^q) \text{ modulo } l.$$

Thus the set  $S$  of residues  $a_q$  modulo  $l$  can be obtained by restricting our attention to the  $l$ -torsion points of  $E$  for each  $l$ . Using those points only, the above comparison will yield each residue  $[a_q]$  modulo  $l$ .

## 4 Torsion polynomials

In order to work with all points in  $E[l]$  simultaneously, Schoof saw that the comparison could be carried out in a particular quotient ring  $R_l = \mathbb{F}_q[x, y]/(f - y^2, f_l)$  of  $\mathbb{F}_q[x, y]/(f - y^2)$ . The torsion polynomials (equivalently called division polynomials), given by

$$\psi_1 = 1,$$

$$\psi_2 = 2y,$$

$$\psi_3 = 3x^4 + 6Ax^2 + 12Bx - A^2,$$

$$\psi_4 = 4y(x^6 + 5Ax^4 + 20Bx^3 - 5A^2x^2 - 4ABx - 8B^2 - A^3),$$

$$\psi_{2m+1} = \psi_{m+2}\psi_m^3 - \psi_{m-1}\psi_{m+1}^3, \quad \text{for } m \geq 2,$$

$$\psi_{2m} = \left(\frac{\psi_m}{2y}\right) \cdot (\psi_{m+2}\psi_{m-1}^2 - \psi_{m-2}\psi_{m+1}^2), \quad \text{for } m \geq 3.$$

are useful for this application because they satisfy the following properties. Let

$$f_m(x, y) = \begin{cases} \psi_m(x, y), & \text{if } m \text{ is odd;} \\ \psi_m(x, y)/2y, & \text{if } m \text{ is even.} \end{cases}$$

- For all positive integers  $m$ , the polynomial  $\psi_m$  is contained in the polynomial ring  $\mathbb{Z}[A, B, x, y]$ . Furthermore, the polynomial  $f_m$  depends only on  $x$ .



- A point  $P = (x, y) \in \mathbb{F}_q \times \mathbb{F}_q$  is a root of the torsion polynomial  $\psi_m$  if and only if  $P$  is a non-zero  $m$ -torsion point of  $E$  over  $\mathbb{F}_q$ . Similarly,  $x$  is a root of  $f_m$  if and only if  $x$  is the  $x$ -coefficient of such a point  $P$ .
- The multiplication by  $m$  map needed to compute  $a_q$  can be expressed as a rational map in the  $\psi_i$ . In particular

$$[m]P = \left( x - \frac{\psi_{m-1}\psi_{m+1}}{\psi_m^2}, \frac{\psi_{2m}}{2\psi_m^4} \right).$$

- For  $l$  odd, the order of  $\psi_l$  or  $f_l$  is  $\frac{1}{2}(l^2 - 1)$ .

## 5 The quotient ring $R_l = \mathbb{F}_q[x, y]/(f - y^2, f_l)$

Because the roots of  $\psi_l$  in  $\mathbb{F}_q$  are the  $l$ -torsion points of  $E(\mathbb{F}_q)$ , when working with the  $l$ -torsion points of  $E$ , we can perform the comparison


$$(x^{q^2}, y^{q^2}) + [q](x, y) \equiv [a_q](x^q, y^q) \text{ modulo } l$$

in the smaller ring  $R_l$ . So Schoof's algorithm iterates over the integer residues modulo  $l$ , and checks for the equality

$$(x^{q^2}, y^{q^2}) + [q \bmod l](x, y) \equiv [a_q \bmod l](x^q, y^q) \text{ in } R_l.$$

Because we are working modulo  $f - y^2$ , all powers of  $y$  greater than or equal to 2 can be reduced to power 0 or 1 in  $R_l$ . Because the order of  $\psi_l$  is  $\frac{1}{2}(l^2 - 1)$ , powers of  $x$  can be similar reduced so that multiplications and comparisons are done with polynomials of  $x$  degree less than or equal to  $\frac{1}{2}(l^2 - 1)$ , and  $y$  degree less than 2.


## 6 Schoof's algorithm – outline of the steps

We are given a prime power order of a finite field,  $q = p^e$ , and an ptic curve  $E: f = y^2 = x^3 + Ax + B$  over that field. We want to find  $\#E(\mathbb{F}_q) = q + 1 - a_q$ .

1. Choose a smallest set of the first  $n$  prime  $S$  such that each prime  $l$  is coprime to  $q$ , and  $\prod_{l \in S} l > 4\sqrt{q}$ .
2. For  $l = 2$ , calculate  $\gcd(x^q - x, f)$ . If the gcd is 1, set  $a_q \equiv 0$  modulo 2, else  $a_q \equiv 1$  modulo 2.

3. For each odd prime  $l \in S$ :
  - (a) Calculate the  $x$  coordinate of  $(x^{q^2}, y^{q^2}) + [q](x, y)$  in  $R_l$  where  $[q]$  is  $q$  modulo  $l$ .
  - (b) For each residue  $n_l$  modulo  $l$ :
    - i. Calculate the  $x$  coordinate of  $[n_l](x^q, y^q)$  in  $R_l$ .
    - ii. Compare the  $x$  coordinates of  $(x^{q^2}, y^{q^2}) + [q](x, y)$  and  $[n_l](x^q, y^q)$  in  $R_l$ .
    - iii. If they are equal:
      - A. Calculate the  $y$  coordinate of  $(x^{q^2}, y^{q^2}) + [q](x, y)$  modulo  $R_l$
      - B. Calculate the  $y$  coordinate of  $[n_l](x^q, y^q)$  in  $R_l$ .
      - C. Compare the  $y$  coordinates of  $(x^{q^2}, y^{q^2}) + [q](x, y)$  and  $[n_l](x^q, y^q)$  in  $R_l$ .
      - D. If they are equal, set  $a_q \equiv n_l$  modulo  $l$ . Else  $a_q \equiv -n_l$  modulo  $l$ .
      - E. Move to next prime  $l$  until  $S$  exhausted. This prime  $l$  is done.
    - iv. Else, continue to next residue  $n_l$  modulo  $l$ .
  - (c) If all residues modulo  $l$  are exhausted, and there was no  $x$  coordinate match, check if  $q$  is a square modulo  $l$ .
    - i. If not, then set  $a_q \equiv 0$  modulo  $l$ .
    - ii. If so, choose  $w$  so that  $w^2 \equiv q$  modulo  $l$ .
      - A. Calculate the  $x$  coordinate of  $(x^q, y^q) - [w](x, y)$ .
      - B. If  $\gcd(\text{numerator of } x\text{-coordinate}, \psi_l) = 1$ , set  $a_q \equiv 0$  modulo  $l$ .
      - C. Otherwise, calculate the  $y$  coordinate. If  $\gcd(\text{numerator of } (y\text{-coordinate})/y, \psi_l) \neq 1$ , set  $a_q \equiv 2w$  modulo  $l$ . Else set  $a_q \equiv -2w$  modulo  $l$ .
4. When all residues of  $a_q$  modulo prime in  $S$  are computed, compute  $a_q$  such that  $a_q$  is the unique integer satisfying those congruences, and in the range  $-2\sqrt{q} \leq a_q \leq 2\sqrt{q}$ .

## 7 Elkies & Atkin improvements

For the Elkies  Atkins improvements to Schoof's original algorithm, we let  $E$  be an elliptic curve defined over  $\mathbb{F}_p$  where  $p$  is a large prime. We also require that  $E$  is

not supersingular, meaning that  $E[p]$  is not trivial. This is not a severe restriction because in the case that  $E/\mathbb{F}_p$  was supersingular, we would have  $\#E(\mathbb{F}_p) = p + 1$ . We also require that  $j(E)$ , the  $j$ -invariant of  $E$ , is not zero or 1728.

The improvements follow from the determination of whether  $l$  is an Elkies or Atkin prime, for each prime  $l \in S$  as in Schoof's. This equates to whether the reduced characteristic polynomial of the Frobenius endomorphism,

$$\chi_l(x) = x^2 - [a_p]x + [p]$$

splits in  $\mathbb{F}_p$ , where  $[a_p]$  and  $[p]$  are  $a_p$  and  $p$  modulo  $l$  respectively, with  $\tau$  the  $p$ -power Frobenius endomorphism and  $a_p$  the trace of the Frobenius as in the  $q$  case. This in turn equates to the question of whether the Frobenius discriminant  $\Delta_{\chi_l} = [a_p]^2 - 4[p]$  is a square in  $\mathbb{F}_l$ .

As  $a_q$  is the quantity to be determined by the algorithm, this cannot be computed at the outset, but is determined from the behaviour of the  $l$ -th modular polynomial  $\Phi_l(x, j(E))$  on  $E$ , described in the next section.

The basis for the performance improvements of Elkies' & Atkin's contributions to Schoof's algorithm are several-fold. For Elkies primes, arithmetic can be carried out in a smaller quotient ring  $R'_l$ , and the search for  $[a_p]$  modulo  $l$  is simplified. The ring will be  $R'_l = \mathbb{F}_q[x, y]/(f - y^2, F_l)$  where  $F_l$  is a degree  $(l - 1)/2$  polynomial, versus the degree  $(l^2 - 1)/2$  degree polynomial  $f_l$ . For Atkin primes, we will need to work with elements in the extension  $\mathbb{F}_{l^2}$  of  $\mathbb{F}_l$ . But the search for the residue  $[a_p]$  modulo  $l$  will only have to consider operations on the primitive  $r$ -th roots of unity in  $\mathbb{F}_{l^2}$  where  $r$  is the degree of the Frobenius endomorphism acting on  $E[l]$ .

## 8 Background to define the modular polynomial

To define the modular polynomial  $\Phi_l(x, j(E))$  and explain its use in SEA, we define the  $j$ -invariant  $j(E)$  of an elliptic curve  $E$ , the Weierstrass  $\wp$  function, and explain the relationship between an elliptic curve  $E/\mathbb{C}$  and the associated lattice  $\Lambda \subset \mathbb{C}$ . We try to keep this minimal and focus on the methods of the algorithms.

Briefly, there is a bijective relationship between isomorphism classes of elliptic curves over  $\bar{K}$ , and the values of the  $j$ -invariant  $j(E)$ , so that  $E_1/\bar{K} \simeq E_2/\bar{K} \iff$

$j(E_1) = j(E_2)$ . The  $j$ -invariant can be defined as  $j(E) = 1728(\frac{4A^3}{4A^3+27B^2})$ . Additionally, each isomorphism class of elliptic curves over  $\mathbb{C}$  is associated with a particular lattice (fully identified by  $\tau = \omega_1/\omega_2 \in \mathcal{H}$  the upper half plane, with  $(\omega_1, \omega_2)$  a homogeneous basis so that  $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \mathbb{Z} + \tau\mathbb{Z}$ )  $\Lambda$  of  $\mathbb{C}$ . With  $E$  in short Weierstrass form, there is a bijective correspondence between  $E$  and  $\mathbb{C}/\Lambda$  given by the map

$$\begin{aligned} \mathbb{C}/\Lambda &\longrightarrow E \\ z + \Lambda &\longmapsto \begin{cases} (\wp(z), (\wp'(z)/2)), & \text{for } z \notin \Lambda; \\ \mathcal{O}, & \text{for } z \in \Lambda. \end{cases} \end{aligned}$$

Here,  $\wp$  is the Weierstrass  $\wp$  function, relative to  $\Lambda$ , given by the series

$$\wp(z; \Lambda) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left( \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right).$$

When the lattice  $\Lambda$  is fixed, as when dealing with a particular elliptic curve, we write  $\wp(z)$ . Furthermore, because of the correspondence between  $j$ -invariants of isomorphism classes of elliptic curves over  $\mathbb{C}$ , and lattices in the complex plane, the  $j$ -invariant function can be given in terms of the lattice  $\Lambda$  in the complex plane, independent of any specific elliptic curve.

Schoof [3] creates the following formal power series in  $\mathbb{Z}[[q]]$  and uses a relation of Jacobi to express  $j$  as a function of  $q = e^{2\pi i\tau}$ .

$$E_4(q) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n} \equiv -48A \text{ modulo } \mathfrak{B}$$

$$E_6(q) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1 - q^n} \equiv 864B \text{ modulo } \mathfrak{B}$$

$$j(q) = 1728 \left( \frac{E_4(q)^3}{E_4(q)^3 - E_6(q)^2} \right).$$

Here  $\mathfrak{B}$  is a prime ideal of  $\mathcal{O}_k$  for a number field  $K$  in which  $E_4(q)$  and  $E_6(q)$  are integers, and the residue field  $\mathcal{O}_k/\mathfrak{B} \simeq \mathbb{F}_p$ . With  $j$  expressed in terms of the complex variable  $\tau \in \mathcal{H}$ , we can define the modular polynomial.

## 9 The modular polynomial $\Phi_l(x, j(E))$

For  $n \in \mathbb{Z}_{>0}$ , let

$$S_n^* = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \text{ such that } a, b, \text{ and } d \in \mathbb{Z}, 0 \leq b < d, ad = n, \text{ and } \gcd(a, b, d) = 1. \right\}$$

Define  $j \circ \alpha$  by  $j \circ \alpha(\tau) = j\left(\frac{a\tau+b}{d}\right)$ .

**Definition 1** (Modular polynomial). *Let  $l \in \mathbb{Z}_{>0}$ . Then the  $l$ -th modular polynomial  $\Phi_l(x, j)$  is given by*

$$\Phi_l(x, j) = \prod_{\alpha \in S_n^*} (x - j \circ \alpha).$$

This  $\Phi_l(x, j)$  has the property that if  $j_{E_1}$  and  $j_{E_2}$  are the  $j$ -invariants of two elliptic curves  $E_1$  and  $E_2$  defined over  $\mathbb{C}$ , then  $\Phi_l(j_{E_1}, j_{E_2}) = 0$  if and only if there is no isogeny of degree  $l$  from  $E_1$  to  $E_2$ .

Let  $E$  be an elliptic curve defined over the finite field  $\mathbb{F}_p$  and  $l$  be a prime coprime to  $p$ . Recall that  $E[l] \simeq \mathbb{Z}/l\mathbb{Z} \times \mathbb{Z}/l\mathbb{Z}$ , and that therefore  $E[l]$  has  $l+1$  cyclic subgroups of order  $l$ . In this situation, the zeroes  $\tilde{j}$  of  $\Phi_l(x, j(E)) = 0$  are the  $j$ -invariants of the isogenous curves  $\tilde{E} = E/C$  where  $C$  is one of those order  $l$  subgroups.

## 10 Distinguishing Atkin & Elkies primes

Recall that a prime  $l$  was an Elkies prime for an elliptic curve  $E$  defined over  $\mathbb{F}_p$  if the characteristic polynomial of the reduced  $p$ -power Frobenius endomorphism  $x^2 - [a_p]x + p = 0$  splits into linear factors over  $\mathbb{F}_l$ , and an Atkin prime otherwise. The modular polynomial provides a way to determine if this polynomial splits without knowing  $a_p$ .

The comparison turns out to be straightforward. If the degree of  $\gcd(\Phi_l(x, j(E)), x^p - x) = 0$ , then  $l$  is an Atkin prime. Otherwise,  $l$  is an Elkies prime. Recall that the roots of  $x^p - x$  are the elements of  $\mathbb{F}_p$ , so that the above gcd will be 1 if and only if  $\Phi_l(x, j(E))$  has no root in  $\mathbb{F}_p$ . In this case, the degree is zero.

The correspondence follows from a theorem of Atkin [3] classifying the possible factorizations of  $\Phi_l(x, j(E))$  in  $\mathbb{F}_p[x]$ . In summary, when  $E/\mathbb{F}_p$  is ordinary with  $j \neq$



0, 1728, and  $\Phi_l(x, j(E)) = h_1 h_2 \dots h_s$  is the factorization of  $\Phi_l(x, j(E))$ , then the  $h_i$  have degree either

1.  $(1, 1, \dots, 1)$  or  $(1, l)$ . In these cases,  $[a_q]^2 - 4p \equiv 0$  modulo  $l$  and so is a square.
2.  $(1, 1, r, \dots, r)$ . In this case  $[a_q]^2 - 4p$  is a square modulo  $l$ .
3.  $(r, r, \dots, r)$ . In this case,  $[a_q]^2 - 4p$  is not a square modulo  $l$ .

## 11 Elkies primes

When  $\Delta_{\chi_l}$  is a square modulo  $l$ , and  $\chi_l(x) = x^2 - [a_p]x + [p] = (x - \lambda)(x - \mu)$  splits in  $\mathbb{F}_l$ , we can find a factor  $F_l$  of division polynomial  $f_l$  with linear degree  $(l+1)/2$ . This new polynomial can be used to create a smaller quotient ring  $R'_l$  in which to find  $[a_p]$  modulo  $l$  such that  $(x^{q^2}, y^{q^2}) + [p](x, y) = [a_p](x^q, y^q)$  as in Schoof's original algorithm.

To construct the polynomial  $F_l$ , first a root of the modular polynomial  $\Phi_l(x, j(E)) \in \mathbb{F}_p[x]$  is found, giving an isogenous curve  $\tilde{E}$  to  $E$ . (In practice, polynomials which have smaller coefficients than the modular polynomials, but have similar properties, such as Müller's modular polynomial  $G_l(x, y)$  [1].) Rarely, this curve may not provide the necessary isogeny and another curve may need to be used, if  $(j, \tilde{j})$  is a singular point of  $\Phi_l(x, y)$ .

From the Weierstrass equations of  $E$  and  $\tilde{E}$ , the coefficients  $a_i$  of

$$F_l(x) = x^{(l-1)/2} + a_{(l-3)/2}x^{(l-3)/2} + \dots + a_0$$

can be computed, using the Laurent series of  $\wp$  as described in [3].

Furthermore, since  $\chi_l(x)$  splits in  $\mathbb{F}_p$ , we know that  $[a_p] = \lambda + \mu = \lambda + \frac{[p]}{\lambda}$  for some  $\lambda, \mu \in \mathbb{F}_p$ , so it suffices to find  $\lambda$ . A theorem of Atkin (treated briefly in next section) categorizes the possible cases for Elkies primes. Both  $\lambda$  and  $\mu$  are the eigenvalues of the Frobenius  $\tau$ , so that there is a point  $P \in E[l] \setminus \mathcal{O}$  with  $\tau(P) = [\lambda]P$ . Expanding the multiplication by  $\lambda$  map, we have that the  $x$ -coefficient of  $P$  must satisfy

$$x^p = x - \frac{\psi_{\lambda-1}\psi_{\lambda+1}}{\psi_\lambda^2}.$$

Since  $P$  is an  $l$ -torsion point, this means that  $P$  satisfies both  $F_l(P) = 0$  and  $\psi_\lambda^2(x^-x) + \psi_{\lambda^{-1}}\psi_{\lambda+1} = 0$ . So the computation of  $[a_p]$  modulo  $l$  is reduced to finding  $\lambda \in \mathbb{F}_l$  such that

$$\gcd(\psi_\lambda^2(x^-x) + \psi_{\lambda^{-1}}\psi_{\lambda+1}, F_l) \neq 1.$$

## 12 Atkin primes

In the case that  $\chi_l(x) = x^2 - [a_p]x + p$  does not split in  $\mathbb{F}_p$ , ie where  $\Phi_l(x, j(E))$  is irreducible in  $\mathbb{F}_p[x]$ , we use another technique. We find the degree  $r$  of the smallest extension field  $\mathbb{F}_{p^r}$  of  $\mathbb{F}_p$  containing the roots of  $\Phi_l(x, j(E))$ . From Atkin's theorem [3],  $r$  will be the smallest integer such that

$$\gcd(\Phi_l(x, j(E)), x^{p^r} - x) = \Phi_l(x, j(E)).$$

Atkin's earlier theorem on the factorization of  $\Phi_l(x, j(E))$  restricts the values that must be checked, as  $r$  must satisfy  $r|(l+1)$  and  $(-1)^{(l+1)/r} = (\frac{p}{l})$ . Further, this  $r$  will equal the degree of the Frobenius endomorphism acting on  $E[l]$ .

Since  $\mathbb{F}_{l^2}$  contains a primitive  $r$ -th roots of unity for  $\mathbb{F}_l$ , it turns out that  $\chi_l(x) = (x - \lambda)(x - \mu)$  splits in  $\mathbb{F}_{l^2}$ , which is isomorphic to  $\mathbb{F}_l[\sqrt{d}]$  for some non-square  $d \in \mathbb{F}_l$ , and that  $\lambda/\mu$  must be such a primitive  $r$ -th root. If  $g$  is a generator of  $\mathbb{F}_{l^2}^*$ , then  $\gamma = g^{(l^2-1)/2}$  is a primitive  $r$ -th root, and powers (all powers  $n$  coprime to  $r$ ) give the other primitive  $r$ -th roots.

Since  $\lambda\mu = [p] \in \mathbb{F}_l$  and  $\lambda + \mu = [a_p] \in \mathbb{F}_l$ , we know that  $\lambda = a_1 + a_2\sqrt{d}$  and  $\mu = a_1 - a_2\sqrt{d}$  for some  $a_1, a_2 \in \mathbb{F}_l$ , and  $d$  non-square as above. After fixing such a  $d$ , Atkin's technique checks for each of the  $\gamma^n = g_{n_1} + g_{n_2}\sqrt{d}$  whether  $p(g_{n_1} + 1)/2$  is a square in  $\mathbb{F}_l$ . If not, that  $\gamma^n$  can be discarded because it cannot possibly satisfy

$$\begin{aligned} \gamma^n = g_{n_1} + g_{n_2}\sqrt{d} &= \frac{\lambda}{\mu} = \frac{\lambda^2}{\lambda\mu} = \frac{a_1^2 + da_2^2 + 2a_1a_2\sqrt{d}}{p} \\ &= \frac{a_1^2 + da_2^2}{p} + \frac{2a_1a_2}{p}\sqrt{d}. \end{aligned}$$

For a particular  $\gamma^n$  to satisfy the above ( $\gamma^n = \lambda/\mu$ ), since  $p = a_1^2 - da_2^2$  and  $pg_{n_1} = a_1^2 + da_2^2$  we would have  $a_1^2 = p(g_{n_1} + 1)/2$ . For the non-discarded  $\gamma^n$ , each  $\pm 2a_1$  is added to the possible  $a_p$  since  $a_p = \lambda + \mu = 2x_1$ . In this technique, a (small) set of possible residues  $[a_p]$  modulo  $l$  is collected, instead of a single value as with Schoof's original algorithm and Elkies primes.

## 13 SEA algorithm – outline of the steps

We are given a prime order of a finite field,  $p$ , and an Elliptic curve  $E: f = y^2 = x^3 + Ax + B$  over that field. We want to find  $\#E(\mathbb{F}_p) = p + 1 - a_p$ . For each Elkies prime, we will keep a residue  $E_l \equiv a_p$  modulo  $l$ . For each Atkin prime, we will keep a set  $A_l$  of possible residues of  $a_p$  modulo  $l$ .

1. Compute the  $j$ -invariant  $j = j(E)$ .
2. Loop over primes  $l$  while  $a_p$  is not fully determined. For each prime  $l$ :
  - (a) Compute  $\gcd(\Phi_l(x, j), x^p - x)$ .
  - (b) If the degree of the gcd is 0, this is an Atkin prime.
    - i. Find degree  $r$  of  $p$ -power Frobenius  $\tau$  acting on  $E[l]$ .
    - ii. Choose a non-square element  $d$  of  $\mathbb{F}_l$ .
    - iii. Find a generator  $g$  of  $\mathbb{F}_l^*$ .
    - iv. Create  $T = \{g^n: \gcd(n, r) = 1, n \in \mathbb{F}_l\}$ .
    - v. For each  $\gamma \in T$ :
      - A. Express  $\gamma$  as  $g_1 + g_2\sqrt{d}$ .
      - B. Check if  $p(g_1 + 1)/2$  is a square in  $\mathbb{F}_l$ . If not, move to next element of  $T$ . If so, calculate  $a_1$  such that  $a_1^2 = p(g_1 + 1)/2$  and add  $\{\pm 2a_1\}$  to the set  $A_l$ , possible residues  $a_p \pmod l$ .
  - (c) Otherwise, this is an Elkies prime.
    - i. Find polynomial  $\mathbb{F}_l$  factor of  $f_l$ .
    - ii. Find  $\lambda \in \mathbb{F}_l$  such that  $\gcd(\psi_\lambda^2(x-x) + \psi_{\lambda-1}\psi_{\lambda+1}, f_l) \neq 1$ .
    - iii. Save  $[a_p] = \lambda + p/\lambda$  as  $E_l$ .
3. Recover  $a_q$  from the  $A_l$  and  $E_l$  residues, as the unique integer satisfying those congruences in the range  $-2\sqrt{q} \leq a_q \leq 2\sqrt{q}$ .

## 14 Complexity of the algorithms and a few benchmarks

Schoof's original algorithm is not implemented in practice, because its  $O(\log^8 q)$  complexity is prohibitive. Using SEA, it may be necessary to work with a larger set of

primes than in Schoof's algorithm, due to the set of possible residues  $[a_p]$  modulo  $l$  when  $l$  is an Atkin prime. However, in SEA the modular polynomials can be precomputed, and with Elkies primes, it is much faster to compute in the smaller quotient ring  $R'_l$ .

The following table is of the algorithm complexities for baby-step / giant step, Schoof's algorithm, and SEA.

	Algorithm		
	BSGS	Schoof's	SEA
Largest prime		$O(\log q)$	$O(\log q)$
Outer loop on $l$		$O(\log q)$	$O(\log q)$
Inner loop over $n_l$		$O(\log^6 q)$ bit ops in $R_l$	$O(\log^4 q)$ bit ops in $R'_l$
Total	$O(q^{1/4})$	$O(\log^8 q)$	$O(\log^6 q)$

SEA is implemented in the PARI number theory software package, which is included with the open source Sage software system. Another common but commercial implementation of SEA is in the Magma computer algebra system. The following rough benchmarks were done in Sage 4.6 (with PARI 2.4.3) and in Magma version 2.17-1 via an online calculator. In each case, the cardinality of  $E : y^2 = x^3 + x + 1$  was found over  $\mathbb{F}_p$  where  $p$  was the first prime with  $n$  digits.

Number of digits of $p$	Algorithm & Implementation		
	BSGS, Sage	SEA, Sage	SEA, Magma
15	1.02 s	4.43 ms	340 ms
20	23.4 s	38.6 ms	370 ms
80	-	7.79 s	8.429 s
90	-	14.9 s	14.720 s
100	-	20.1 s	19.339 s
110	-	24.8 s	26.600 s
120	-	54.2 s	48.649 s
130	-	fail	59.659 s
140	-	fail	> 60 s, so could not complete

Because these tests were run on different computers, with different hardware con-

figurations and different operating systems, they cannot be taken as a fine-grained comparison. However, they clearly highlight the improvement that Schoof's algorithm (and SEA in particular) were to previous algorithms.

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