A summary of Classical Lamination Theory

Defining the Laminate

A laminate is an organized stack of uni-directional composite plies (uni-directional meaning the plies have a single fiber direction rather than a weave pattern). The stack is defined by the fiber directions of each ply like this:

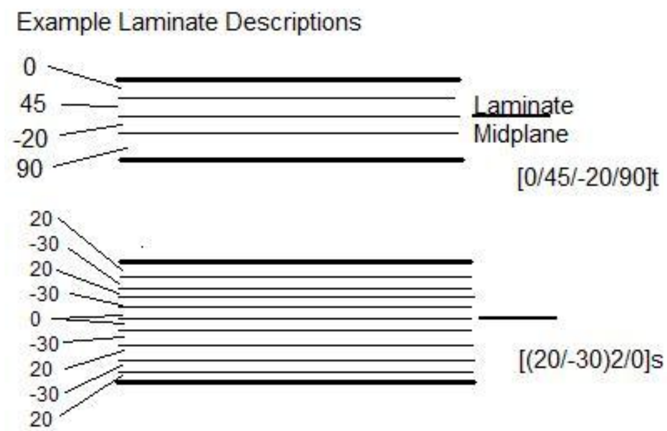

Figure 2. Example laminate stacking sequences. Note that the t, s, and 2 outside of the parenthesis should be subscript.

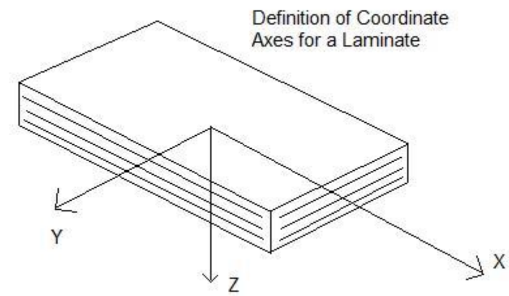

Figure 1. Definition of general coordinate axes.

The t stands for "truncate," the s for "symmetrical" (implying the listed sequences should be mirrored across the laminate's midplane) and the 2 outside of the parenthesis means that sequence is repeated twice. The fiber angles are measured from a general coordinate system defined in figure 2. (Note that the positive z axis points down.)

While the whole laminate is defined according to this x-y-z coordinate system, in an individual ply, the "11" direction indicates the fiber direction, and the "22" direction is normal to the fiber direction.

Material Properties

In addition to the stacking sequence of the laminate, the material properties of the composite material must be defined. The following properties must be defined:

- Mechanical Elasticity (E_{11}, E_{22}, G_{12}, and υ_{12})
- Environmental Elasticity (α_{11} , α_{22} , β_{11} , β_{22}) which represent thermal and moisture expansion, respectively.

Mechanical and Environmental Loads

Finally, the mechanical and Environmental loads must be defined:

- Normal Forces (N_{xx}, N_{yy}, N_{xy})
- Moment (twisting) forces (M_{xx}, M_{yy}, M_{xy})
- Environmental (ΔT and ΔM in Celsius and % Moisture, respectively)

CLT Calculations – the ABD Matrix

The ABD matrix is a 6x6 matrix that serves as a connection between the applied loads and the associated strains in the laminate. It essentially defines the elastic properties of the entire laminate. To assemble the ABD matrix, follow these steps:

1. Calculate reduced stiffness matrix **Q**$_{ij}$ for each material used in the laminate (if a laminate uses only one type of composite material, there will be only 1 stiffness matrix). The stiffness matrix describes the elastic behavior of the ply in plane loading

$$\boldsymbol{Q}_{ij} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix}$$

Where

$$Q_{11} = \frac{E_{11}^2}{(E_{11} - \upsilon_{12} \cdot E_{22})}, \quad Q_{12} = \frac{\upsilon_{12} E_{11} E_{22}}{E_{11} - \upsilon_{12}^2 E_{22}}$$

$$Q_{22} = \frac{E_{11} E_{22}}{E_{11} - \upsilon_{12}^2 E_{22}}, \quad Q_{66} = G_{12}$$

2. Calculate the transformed reduced stiffness matrix $\overline{Q_{ij}}$ for each ply based on the reduced stiffness matrix and fiber angle.

Where

$$\overline{Q_{11}} = Q_{11}\cos(\theta)^4 + 2(Q_{12} + 2Q_{66})\cos(\theta)^2 \cdot \sin(\theta)^2 + Q_{22}\sin(\theta)^4$$
$$\overline{Q_{12}} = \overline{Q_{21}} = Q_{12}(\cos(\theta)^4 + \sin(\theta)^4) + (Q_{11} + Q_{22} - 4Q_{66})\cos(\theta)^2\sin(\theta)^2$$
$$\overline{Q_{16}} = \overline{Q_{61}} = (Q_{11} - Q_{12} - 2Q_{66})\cos(\theta)^3\sin(\theta) - (Q_{22} - Q_{12} - 2Q_{66})\cos(\theta)\sin(\theta)^3$$
$$\overline{Q_{22}} = Q_{11}\sin(\theta)^4 + 2(Q_{12} + 2Q_{66})\cos(\theta)^2\sin(\theta)^2 + Q_{22}\cos(\theta)^4$$
$$\overline{Q_{26}} = \overline{Q_{62}} = (Q_{11} - Q_{12} - 2Q_{66})\cos(\theta)\sin(\theta)^3 - (Q_{22} - Q_{12} - 2Q_{66})\cos(\theta)^3\sin(\theta)$$
$$\overline{Q_{66}} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})\cos(\theta)^2\sin(\theta)^2 + Q_{66}(\cos(\theta)^4 + \sin(\theta)^4)$$

$$\overline{\boldsymbol{Q}_{ij}} = \begin{matrix} \overline{Q_{11}} & \overline{Q_{12}} & \overline{Q_{16}} \\ \overline{Q_{12}} & \overline{Q_{22}} & \overline{Q_{26}} \\ \overline{Q_{16}} & \overline{Q_{26}} & \overline{Q_{66}} \end{matrix}$$

3. Calculate the ***A***$_{ij}$, ***B***$_{ij}$, ***D***$_{ij}$ matrices using the following equations where z represents the vertical position in the ply from the midplane measured in meters:

$$\boldsymbol{A}_{ij} = \sum_{k=1}^{n} \{Q_{ij}\}_n (z_k - z_{k-1})$$
$$\boldsymbol{B}_{ij} = \frac{1}{2}\sum_{k=1}^{n} \{Q_{ij}\}_n (z_k^2 - z_{k-1}^2)$$
$$\boldsymbol{D}_{ij} = \frac{1}{3}\sum_{k=1}^{n} \{Q_{ij}\}_n (z_k^3 - z_{k-1}^3)$$

4. Assemble ABD:

$$\boldsymbol{ABD} = \begin{bmatrix} A & B \\ B & D \end{bmatrix}$$

5. Calculate inverse of **ABD**: **abd** = **ABD**$^{-1}$

6. Calculate thermal and moisture expansion coefficients for each ply:

   Calculate the effective thermal and moisture expansion coefficients for each ply:

$$\begin{gathered}
\propto_{xx} = \propto_{11} \cos(\theta)^2 + \propto_{22} \sin(\theta)^2 \\
\propto_{yy} = \propto_{11} \sin(\theta)^2 + \propto_{22} \cos(\theta)^2 \\
\propto_{xy} = 2\cos(\theta)\sin(\theta)\,(\propto_{11} - \propto_{22})
\end{gathered}$$

$$\begin{gathered}
\beta_{xx} = \beta_{11} \cos(\theta)^2 + \beta_{22} \sin(\theta)^2 \\
\beta_{yy} = \beta_{11} \sin(\theta)^2 + \beta_{22} \cos(\theta)^2 \\
\beta_{xy} = 2\cos(\theta)\sin(\theta)\,(\beta_{11} - \beta_{22})
\end{gathered}$$

   Calculate thermal and moisture stress and moment resultants:

   Thermal Resultants:

$$N_{xx}^T = \Delta T \sum_{k=1}^{n} \left\{ [\overline{Q_{11}}\alpha_{xx} + \overline{Q_{12}}\alpha_{yy} + \overline{Q_{16}}\alpha_{xy}]_k [z_k - z_{k-1}] \right\}$$

$$N_{yy}^T = \Delta T \sum_{k=1}^{n} \left\{ [\overline{Q_{12}}\alpha_{xx} + \overline{Q_{22}}\alpha_{yy} + \overline{Q_{26}}\alpha_{xy}]_k [z_k - z_{k-1}] \right\}$$

$$N_{xy}^T = \Delta T \sum_{k=1}^{n} \left\{ [\overline{Q_{16}}\alpha_{xx} + \overline{Q_{26}}\alpha_{yy} + \overline{Q_{66}}\alpha_{xy}]_k [z_k - z_{k-1}] \right\}$$

$$M_{xx}^T = \frac{\Delta T}{2} \sum_{k=1}^{n} \left\{ [\overline{Q_{11}}\alpha_{xx} + \overline{Q_{12}}\alpha_{yy} + \overline{Q_{16}}\alpha_{xy}]_k [z_k^2 - z_{k-1}^2] \right\}$$

$$M_{yy}^T = \frac{\Delta T}{2} \sum_{k=1}^{n} \left\{ [\overline{Q_{12}}\alpha_{xx} + \overline{Q_{22}}\alpha_{yy} + \overline{Q_{26}}\alpha_{xy}]_k [z_k^2 - z_{k-1}^2] \right\}$$

$$M_{xy}^T = \frac{\Delta T}{2} \sum_{k=1}^{n} \left\{ [\overline{Q_{16}}\alpha_{xx} + \overline{Q_{26}}\alpha_{yy} + \overline{Q_{66}}\alpha_{xy}]_k [z_k^2 - z_{k-1}^2] \right\}$$

Moisture Resultants:

$$N_{xx}^{M} = \Delta T \sum_{k=1}^{n} \{[\overline{Q_{11}}\beta_{xx} + \overline{Q_{12}}\beta_{yy} + \overline{Q_{16}}\beta_{xy}]_k [z_k - z_{k-1}]\}$$

$$N_{yy}^{M} = \Delta T \sum_{k=1}^{n} \{[\overline{Q_{12}}\beta_{xx} + \overline{Q_{22}}\beta_{yy} + \overline{Q_{26}}\beta_{xy}]_k [z_k - z_{k-1}]\}$$

$$N_{xy}^{M} = \Delta T \sum_{k=1}^{n} \{[\overline{Q_{16}}\beta_{xx} + \overline{Q_{26}}\beta_{yy} + \overline{Q_{66}}\beta_{xy}]_k [z_k - z_{k-1}]\}$$

$$M_{xx}^{M} = \frac{\Delta T}{2} \sum_{k=1}^{n} \{[\overline{Q_{11}}\beta_{xx} + \overline{Q_{12}}\beta_{yy} + \overline{Q_{16}}\beta_{xy}]_k [z_k^2 - z_{k-1}^2]\}$$

$$M_{yy}^{M} = \frac{\Delta T}{2} \sum_{k=1}^{n} \{[\overline{Q_{12}}\beta_{xx} + \overline{Q_{22}}\beta_{yy} + \overline{Q_{26}}\beta_{xy}]_k [z_k^2 - z_{k-1}^2]\}$$

$$M_{xy}^{M} = \frac{\Delta T}{2} \sum_{k=1}^{n} \{[\overline{Q_{16}}\beta_{xx} + \overline{Q_{26}}\beta_{yy} + \overline{Q_{66}}\beta_{xy}]_k [z_k^2 - z_{k-1}^2]\}$$

7. Calculate midplane strains and curvatures induced in the laminate. These represent the deflections of the laminate.

$$\begin{bmatrix} \varepsilon_{xx}^0 \\ \varepsilon_{yy}^0 \\ \varepsilon_{xy}^0 \\ \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{16} & b_{11} & b_{12} & b_{16} \\ a_{12} & a_{22} & a_{26} & b_{12} & b_{22} & b_{16} \\ a_{16} & a_{26} & a_{66} & b_{16} & b_{26} & b_{66} \\ b_{11} & b_{12} & b_{16} & d_{11} & d_{12} & d_{16} \\ b_{12} & b_{22} & b_{26} & d_{12} & d_{22} & b_{26} \\ b_{16} & b_{26} & b_{66} & d_{16} & d_{26} & d_{66} \end{bmatrix} \cdot \begin{bmatrix} N_{xx} + N_{xx}^T + N_{xx}^M \\ N_{yy} + N_{yy}^T + N_{yy}^M \\ N_{xy} + N_{xy}^T + N_{xy}^M \\ M_{xx} + M_{xx}^T + M_{xx}^M \\ M_{yy} + M_{yy}^T + M_{yy}^M \\ M_{xy} + M_{xy}^T + M_{xy}^M \end{bmatrix}$$

8. For each ply

a. Calculate ply strains in the x-y coordinate system

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx}^0 \\ \varepsilon_{yy}^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z \begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix}$$

b. Calculate ply stresses in the x-y coordinate system

$$\begin{bmatrix} \boldsymbol{\sigma_{xx}} \\ \boldsymbol{\sigma_{yy}} \\ \boldsymbol{\tau_{xy}} \end{bmatrix} = \begin{bmatrix} \overline{Q_{11}} & \overline{Q_{12}} & \overline{Q_{16}} \\ \overline{Q_{12}} & \overline{Q_{22}} & \overline{Q_{26}} \\ \overline{Q_{16}} & \overline{Q_{26}} & \overline{Q_{66}} \end{bmatrix} \cdot \begin{Bmatrix} \boldsymbol{\varepsilon_{xx} - \Delta T \alpha_{xx} - \Delta M \beta_{xx}} \\ \boldsymbol{\varepsilon_{yy} - \Delta T \alpha_{yy} - \Delta M \beta_{yy}} \\ \boldsymbol{\varepsilon_{xy} - \Delta T \alpha_{xy} - \Delta M \beta_{xy}} \end{Bmatrix}$$