Sundials and Apparent Solar Motion

Math 480B

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Getting Started

In order to understand the astrological motions that allow us to tell time with a sundial, there are some basic concepts that must be understood. Since before Christ, civilizations have known that the earth is round. And thanks to Galileo we know that the sun is the center for our orbit. Most realize that the revolution around the sun is in a counterclockwise manner. Yet the orbit of our planet in not perfectly circular, but instead takes a more elliptic path. But the orbit is not 'perfect' as the sun is not the very center, but instead favors one side of the ellipse.

At the same time Earth revolves around the sun, it also turns about its own axis of rotation. The axis of rotation is tilted, however, by 23.5°. Observers (or even those who care nothing of science) can see that the sun 'rises' in the east and 'sets' in the west - showing that the rotation of our planet is indeed counterclockwise. We make one full rotation about every 24 hours. This number is not exact day by day, but instead an annual mean.

Before we digress any further into the consequences of earth's non-perfect orbit we should explore some facts about the orbit around the sun, and around earth's own axis. The earth is approximately 150 million kilometers away from the sun. This distance varies, however, as the earth follows an elliptical orbit around the sun. Earth is closest to the sun during the winter, at a time in its orbit called the *perihelion*. During the *aphelion* the earth is the furthest from the sun – giving a range of about five million kilometers. Yet the distance the earth travels around the sun is almost six times greater, 940 million kilometers. This orbit takes the earth a little more than 365 days, hence the 'leap year' was introduced to the Gregorian calendar in 1582 AD.

Orbit about the sun		
	Orbit time	365.242199 days
	Orbital speed	30 km/s
	Distance traveled	940,000,000 km
Rotation about the earth's axis		
	Rotation time (mean)	24 hours
	Rotation speed	.45 km/s
Distance from the sun		150,000,000 km
	Average distance	149,597,888 km
	Perihelion	147,098,074 km
	Aphelion	152,097,701 km
	Range	4,999,627 km

Seasons

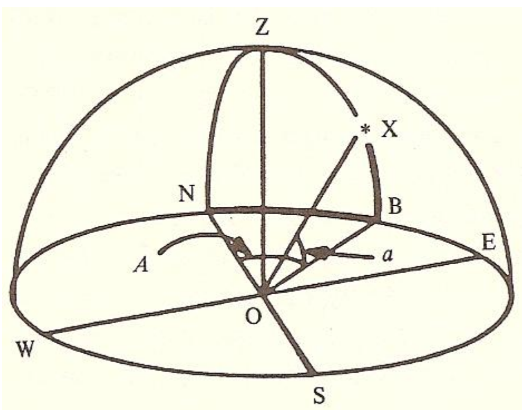
It was mentioned before that the axis of the earth takes on a slight tilt. Though we casually grazed by this fact, it actually is very significant in the explanation of the four seasons we experience. Simply stated, when a part of the earth is tilted towards the sun, it experiences warmer weather, what we call summer. Conversely, winter occurs when the given location is tilted away from the earth. For example, the northern hemisphere experiences the peak of summer in June when the northern part of the axis faces the sun and the southern is pointing away. Winter peaks in December when the northern tip of the axis points away from the sun. Autumn and spring are transitional seasons when the earth's axis is pointing neither directly towards, nor away from the sun. Note that latitudes nears the equator experience less drastic changes in temperature because they do not favor (are not closer to) the northern nor southern axis.

Specific terms have been given to describe the tilt of the earth's axis throughout a given year. When the axis is most inclined towards or away from the sun a *solstice* occurs. At this time, the sun appears most northern or most southern in the sky. Similarly, an *equinox* occurs twice per year. As one would conjecture, this occurs in between the two solstices, particularly when the earth's axis points neither towards nor away from the sun. At an equinox, the zenith as observed from the equator appears to intersect the sun. It is to be noted that the solstices and equinoxes do not occur in consistent increments. This is because of the elliptical orbit that earth follows around the sun. Thus in the northern hemisphere summer and spring last longer than autumn and winter.

Coordinate Systems

The Horizon System

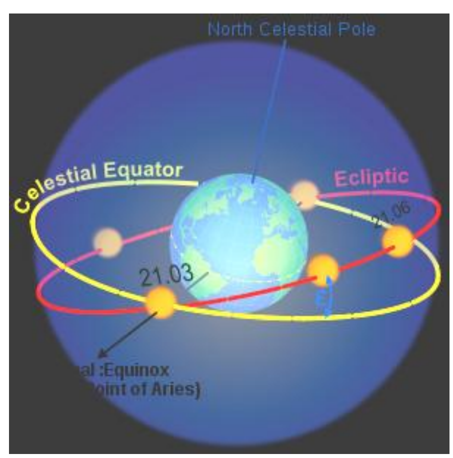
A fixed coordinate system must be established to be able to speak of astrological locations quantitatively. The most natural system mimics our longitude and latitude system for locating positions on Earth. This system is called the *horizontal coordinate system.*



The horizontal coordinate system uses a point of reference, O, on the surface of the Earth as its origin. The line formed by connecting the point representing the center of the Earth to the point O that continues out into space is the *zenith* axis. This axis is normal to the horizon, which can be considered an XY plane, where the y-axis points north and the x-axis points east.

Given this system, assume the sun is visible in the sky. If we point a unit vector to the sun with initial point O, then we see that we can represent this vector by its polar coordinates. However, this astronomical system does not follow common mathematical conventions. Instead of measuring its XY angle counterclockwise from the x-axis (east), we measure the angle clockwise from the y-axis (north). This angle has a special name in astronomy, the *azimuth*, A. The second angle measured is the angle the vector makes with the XY plane (the horizon). This angle is called the *elevation* angle and does correspond to mathematical convention (a in the diagram).

The Ecliptic System



The *ecliptic* coordinate system is one that is centered at the Sun and whose coordinates are based on the plane formed by the orbit of the Earth, i.e. the *ecliptic plane*. The ecliptic is also used to refer to the path that the Sun traces along our sky throughout the year if we measure it every *sidereal* day. This path progresses in a counterclockwise direction until the sun appears in the same position, i.e. a year, or one revolution around the sun, has passed. The ecliptic poles are *Draco* to the north and *Dorado* to the south. Relative to these poles, earth's axis is tilted 23.5°. This means the two poles, and thus entire orbits are separated by 23.5°. This causes the sun to appear to move north to south in the sky.

How Sundials Work

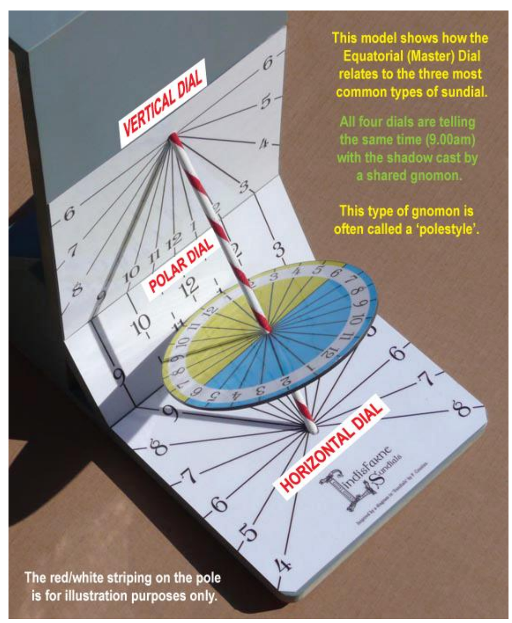
Introduction to Sundials

The basic principle of how a sundial works is very simple. The most general idea is that a shadow is cast onto some sort of surface and from that shadow the correct time can be found. The surface onto which the shadow is cast is referred to as the *dial face* or the *dial plate* and does not have to be a flat surface. On the dial face, it is expected to find hours lines, which map out where the shadow will be according to the hour of the day. The object from which the shadow is formed is called the *gnomon* (noh-mon). Gnomon is derived from the Greek, γνώμων, which means "indicator", "one who discerns," or "that which reveals.” Generally there is a specific part of the shadow that is used to tell time, the part of the gnomon that casts this piece of the shadow is called the *style*. In certain sundial designs time and date are told only by the tip of the shadow given from a sharp feature on the gnomon. The point-like feature that casts this shadow is called the *nodus* and is frequently found on the tip of the style. In most sundials, the style is fixed, but for certain dials it may have a moving feature to adjust for month.

Generally speaking, when we refer to sundials and their mechanics we are referring to sundials in the northern hemisphere. This is largely due to the fact that most sundials can be found in the northern hemisphere. This is not to say that there aren’t sundials in the southern hemisphere, or that they don’t work in the southern hemisphere. In fact, a sundial made for the northern hemisphere will work perfectly fine in the southern hemisphere as long as it is placed at its perspective latitude in the sound and the gnomon is correctly aligned. The main difference is that sundials in the southern hemisphere function somewhat backwards to northern hemisphere dials; instead of the style pointing due north, it would point due south, if the shadow normally moved across the dial face clockwise, it would now move counter-clockwise.

Equatorial Sundial

Equatorial sundials are modeled after the earth, and thus used as a base for building other types of sundials. The model consists of a circular dial plate with a gnomon at the center, positioned perpendicular to the plate. As to imitate the axis of the earth, the gnomon must point toward the celestial poles. The plate is then positioned so that it is parallel to the equator. This is achieved by tilting the dial so that the angle between the gnomon and the (flat) ground is equal to the latitude of the given location. Setup for this type of sundial is thus dependent upon the physical location of the dial.



Earth makes one full rotation every 24 hours. Seeing that earth is round, a single point makes a 360° rotation before returning to its same location from 24 hours before. An equatorial sundial mimics the circularity of the earth, and the gnomon is parallel to the axis of the earth. Thus every degree of rotation experience by the earth is experienced by the sundial to the same degree and the sundial too makes one full 'rotation' (with the earth) in 24 hours. The shadow then shifts a consistent amount, 15° (360°/ 24 hours), every hour. When the sun rises in the east, shadows are casted toward the westerly side of the sundial. As the day progresses, the shadow rotates in a clockwise manner until the sun sets in the west, casting a last shadow on the easterly side of the dial plate. Noon occurs when the sun is highest in the sky, causing the shadow to cast directly towards the ground. Thus 12:00 noon is observed as the lowest point on the dial plate. From this point, the remaining hours of the day can be marked off, clockwise, in 15° increments.

Equatorial sundials read the time on both sides of the circular dial plate. From the vernal equinox to the autumnal, the earth's axis is oriented towards the sun causing light to fall on the northern half of the dial plate. Yet from the autumnal equinox to the vernal equinox the southern axis points towards the sun, and light is cast on the underside of the sundial. To use the sundial during these months, the gnomon must extend to the southern side of the sundial. Also, the numbers on this half of the dial plate run counter-clockwise. But equatorial sundials have a major drawback; during equinoxes sunlight falls directly on the edges of the circular base of the dial plate, and the dial plate is rendered completely useless.

Horizontal Sundial

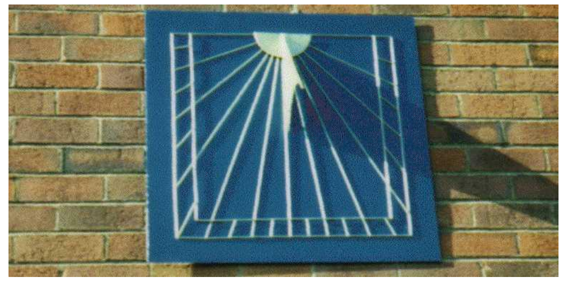
When a person thinks of sundial, the image that normally comes to mind the horizontal sundial, where the dial's face lays flat on the ground. This is the most common type and is also the easiest to read. In a horizontal sundial, because the style is in alignment with the Earth's rotational axis, the style always points true north. Unlike the equatorial dial, the horizontal dial's dial face is not perpendicular to the gnomon and therefore the lines on the dial face are not spaced out evenly to accommodate this difference. Instead the lines are spaced out according to the formula $\tan\theta = \sin\lambda\tan(15°\times t)$. Where λ is the sundial's geographical latitude, θ is the angle between a given hour line and the noon hour line, and t is the number of hours before or after noon.

The horizontal sundial can function all the way up to the North Pole, where due to the style pointing directly upward, the dial would become an equatorial dial. However, when used a the equator, the horizontal sundial becomes impractical because the latitude is 0° rendering the style to be completely flat and hence not casting a shadow at all. One of the chief advantages of the horizontal sundial is that the sun lights the dial face year round, and therefore is a reliable method of telling time year-round. Another great advantage is the ability to use a horizontal dial at more than one latitude. For example, if a sundial were created to tell time at a latitude of 30°

it could be used at 40°. The reader of the sundial would merely have to tilt the sundial 10° so that the style still continue to point north.

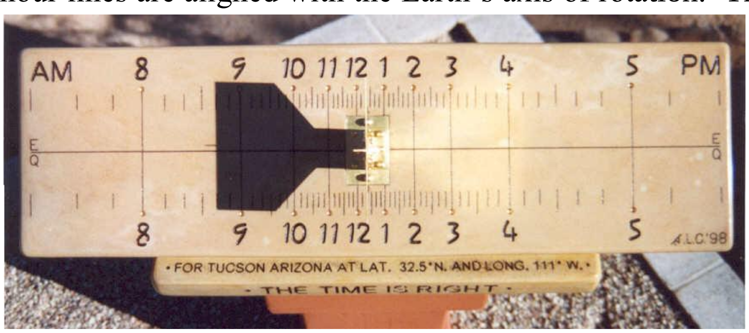
Vertical Sundial

The vertical sundial has construction very similar to the horizontal sundial except that the shadow receiving plane is vertical instead of horizontal. Also different from the horizontal and equatorial sundials, the shadow on a vertical sundial moves in a counter-clockwise direction. Like the horizontal sundial, the vertical dial does not have evenly spaced hour lines, instead the lines are spaced with a near identical formula: $\tan\theta = \cos\lambda\tan(15° \times t)$. Like in the horizontal sundial, λ is the sundial's geographical latitude, θ is the angle between a given hour line and the noon hour line, and t is the number of hours before or after noon. This formula is valid for vertical sundials that face south. A vertical dial the faces exactly north, south, east, or west is called a vertical direct dial. If the dial is not facing south, the hours for which it works are limited: east dials work only in the morning, west in the evening, and north vertical dials function either before 6 am or after 6 pm. Dials that do not face directly in one of these directions are called declining dials. It is mathematically very difficult to calculate where the hour lines should be on a declining dial and is often done by observation instead.



Polar Sundial

The polar sundial has a construction much different than the equatorial, horizontal, and vertical sundials. A polar sundial is very simple to construct and extremely easy to read. With a polar dial, the hour lines on the dial face are parallel to the gnomon/style. As with the gnomon, the hour lines are aligned with the Earth's axis of rotation. Though it is not necessary, the dial plate is most often tilted to the angle of latitude where it is located. Most commonly, the noon hour is placed where the sunlight directly hits the gnomon and no shadow is cast. The hour lines are closest together near the gnomon and spaced furthest apart when the sun is lower on the horizon as the shadow moves more quickly here. The perpendicular spacing X of the hour lines can be obtained from this formula: $X = H\tan(15° \times t)$ where H is the height of the gnomon/style above the plane and t is the number of hours away from the center time (usually noon) on the dial.



A Perfect Day, A Perfect Sundial

Apparent Solar Time

Time as given by a sundial is scientifically known as apparent solar time. The determinant of this measurement is the apparent, or true position of the sun. One solar day occurs when the earth makes a complete rotation and the sun returns to the meridian; an imaginary circle running along the longitudinal lines of earth.

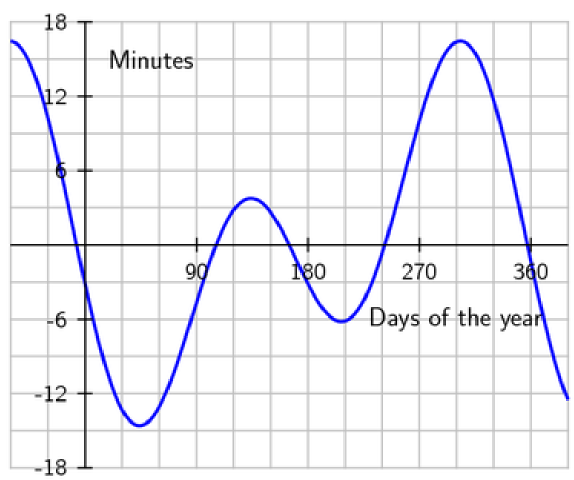
Standard Clock Time

Though sundials were a chief source of time keeping for generations, the need for a consistent measurement of time made apparent solar time untenable. What people needed of time was a consistent length of day, something that apparent solar time did not deliver. The time it takes for the sun to return to the same position in the sky, i.e. what a sundial measures, changes from day to day. (To understand this, see the section below titled *Analemma*). The time system that we use now adjusts for this inconsistency by removing the two causes of apparent solar time's unequal length of days: the angle our equatorial plane makes with the ecliptic and also the ellipticity of the Earth's orbit around the Sun.

However, our system goes one step further. We also incorporate time zones so that a broader range of longitudes share the same time (rather than each longitude having its own time). Time zones came out of a necessity to run trains over longer distances. Additionally, we also practice daylight savings time which adds an hour to our time so that farmers have greater sunlight throughout the day.

To convert solar time to clock time, several adjustments must be made. A first consideration must be taken into the application of daylight saving. If a sundial is read during the spring in a region that upholds daylight saving time, one hour must be added to the dial's reading. Though the practice of daylight saving is controversial and currently used in less than half of the world.

We must also adjust for the 24 time zones. Standard clock time is consistent throughout a time zone, thus an adjustment must be made when converting between solar and standard clock time. An extreme example of how great this adjustment can be occurs in China which upholds one time zone throughout the entire country. The zone includes areas between 142°30′ E and 67°30′ E. In this zone, solar noon can be experienced as late as 15:00 standard clock time. To adjust from solar time to clock time, add four minutes for every degree west from the center of the time zone or subtract four minutes for every degree east.



Finally, to correct for the fact that the Earth is in an elliptic orbit and also has an axis of rotation angled to the ecliptic plane, one must use the *Equation of Time*, which is a universal correction, not based on the latitude of the sundial. The Equation of Time can be incorporated manually to a sundial reading, but can also be included in the sundial itself by curved hour lines.

Sidereal Time

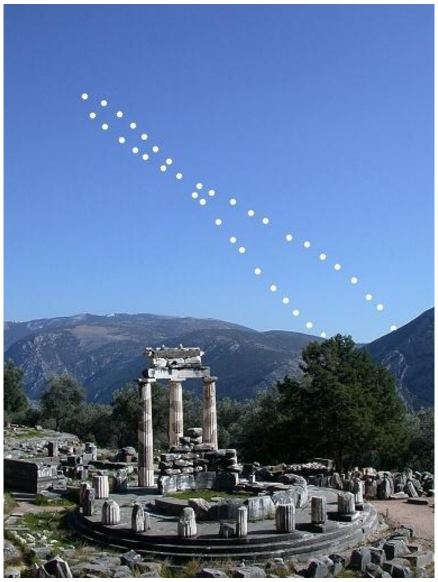
To understand solar time, it is helpful to put it in perspective relative to what is called *sidereal time*. In particular, one will commonly hear of a *sidereal day*. This is the interval of time for a star to return back to the same azimuth. This corresponds to the time that it takes for the Earth to make one complete rotation. The adjective, sidereal, simply means that one is basing their time on the apparent motion of the stars rather than the apparent motion of the sun.

A sidereal day is always shorter than a solar day. This can be explained by the fact that a sidereal day measures the time it takes for the Earth to rotate and face the same direction relative to the ecliptic plane. On the other hand, a solar day is a measure of the time it takes for the earth to face the sun once again. This will always take longer than a sidereal day because the earth revolves a bit around its orbit during a day. Due to that very positional change, it must compensate (wait longer) if it wants to point to the sun again.

Though the sidereal day measures the exact time it takes for the Earth to make one complete rotation, it is not a good time interval for our days. To understand why, imagine that we watch a star one night and wait until it reaches its highest point in the sky. Let us consider that exact moment 12 midnight. Now, imagine that we that we continue to measure sidereal days until the Earth travels to the opposite side of its orbit from the day that we first measured. On that day, 12 midnight would occur during daylight because the Earth is now facing the sun in that direction. Obviously, to have a time completely reverse from night to daylight and vice versa would be useless to our lives that depend on daylight hours.

Analemma

The analemma is a natural phenomenon that occurs due to the Earth's orbital eccentricity and its tilt. If one measures the sun's azimuth and elevation at the same *mean solar time* each day throughout the year, the sun will appear to form a figure eight in the sky. This amounts to measuring the same time on your watch, not observing daylight savings time.



This shape is found on globes and some sundials. What it measures is the relative time difference between *mean solar time* and *local solar time*. Each point on the analemma corresponds to a moment in time where the sun is either at, east, or west of the expected position of the Sun if we did not have an elliptic orbit or a tilted axis of rotation. When the sun is at the same position, then in our local solar day was as long as a mean solar day. When the sun is east of its expected mean position, the local solar day is longer than the mean solar day. When the sun is west of its expected position, the local solar day was shorter than a mean solar day. In the former case, one would need to add a number of minutes to the time when the sun returns back to its azimuth position. In the latter case, time must be subtracted when the sun returns back to its azimuth position.

The first cause of this analemma is that the Earth's orbit around the sun is elliptical with eccentricity 0.017. Thus the earth moves faster at the perihelion, and slower at the aphelion. The Earth's axial tilt is the second cause for the variation in length of a solar day. Because of the tilt, the sun appears to move faster or slower throughout the year because XY velocity components change with time. When the sun crosses the equator at the equinoxes, it is moving at an angle to the equator causing the projection of the motion to be slower than average. This is seen as a shortened solar day. Conversely, days appear longer during the solstices. At this time the sun is moving parallel to the equator causing the projection of the motion to be faster than the mean. Yet the maximum seasonal deviations from the mean solar day is only 16 minutes. Thus 24 hours is generally assumed to be the length of a day on both a sundial and an every day wrist watch.

Mathematics Behind a Sundial and Incorporating Sage

Our model

The apparent angle of the Sun in the horizon coordinate system is the most important information when dealing with sundials and time. For this reason, we started off the project by attempting to model the motion of the Sun through the sky with some equations based on Kepler's laws of planetary motion and conversions from the ecliptic coordinate systems to the horizon coordinate system. We got as far as being able to predict the Earth's location given a time since perihelion (closest the Earth comes to the Sun).

Solving for location involved computing the mean, eccentric, and true anomalies and heliocentric distance from other properties of Earth's orbit. Finding the eccentric anomaly is equivalent to solving the famous Kepler's Equation, whose solution is the topic of many research papers. In Sage, we just used `find_root` to approximate a decimal value! From here, all we had to do was convert from ecliptic coordinates to equatorial then to local horizon. However, we stumbled across a telnet accessible program called Horizons, which gave us a much easier and more accurate way to get our angle data (see below).

Horizons

The NASA Jet Propulsion Laboratory has a system called Horizons that gives extremely accurate ephemeris data (table of values) for most celestial objects, including the Earth and Sun. The system can be connected to via Telnet, and with the help of Pexpect we were able to retrieve azimuth and elevation angles of the sun for any longitude and latitude on Earth and for any recent time. Horizons takes into account variables that we would have never been able to predict with a model we made, including gravitational attraction to other planets in the solar system, difference in light time, and defraction of the Sun's light through the atmosphere.

"Stick" projections

To model sundials in their simplest form, all we really needed was a way to project the endpoints of an arbitrary line segment on an arbitrarily oriented plane. Connecting those two points gives a shadow of any sundial's gnomon on any wall or angled ground we needed.

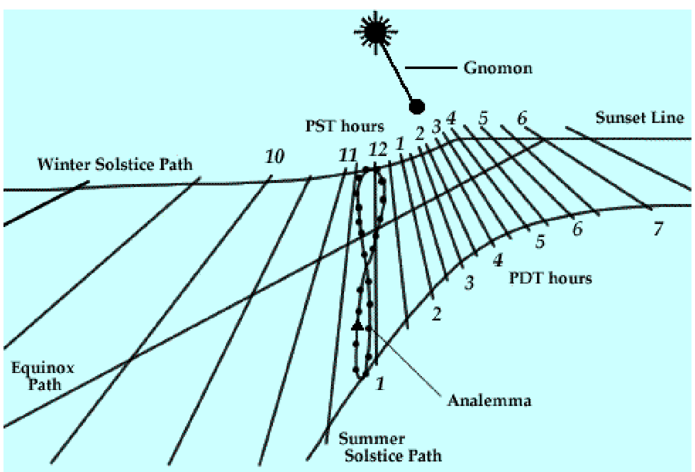
To accomplish this, we found the intersection of the plane and the line connecting each endpoint to the Sun. Those projected points are connected by a line to give the shadow at a certain time. We used this to model all of the different types of sundials, as well as explore projections from sundials we made up. We found some pretty cool results!

Applications

Physics and Astronomy Building

A beautiful example of (and variation of) a vertical sundial can be found right on our University of Washington campus. The southwest facing wall of the Physics and Astronomy building contains a sundial that does more than just tell time. Its gnomon aligned with the northern celestial pole also depicts the analemma, while shadow lengths vary between the summer and winter solstice.

As the sun moves east to west across the wall, the shadow from the gnomon follows a continuous path left to right. When it is solar noon, the sun is due south causing the shadow to be completely vertical. The spacing between hour intervals becomes noticeably longer during the early and late hours in the day. In the summer, the sun is higher causing the shadow length to be longer, reaching its maximum at the summer solstice. A similar line is mapped for the winter solstice when the sun is relatively low, causing the shadow length to reach a minimum. Most interesting is the shadow during the equinoxes, which follows a perfectly straight path. Another interesting depiction on the wall is the analemma formed by the sun at solar noon throughout the year. From the location of the sun on the analemma, an approximate date can be assumed.



We were able to accurately model this sundial by estimating a normal vector relative to north for this particular wall of the astronomy building and projecting the shadow of a stick on this wall aligned with the polar axis. This is best illustrated by Figure 5 and Figure 6 in the table of figures below.

Illumination

With accurate data on the motion of the sun and powerful graphing software at our fingertips, we set out to plot a map of solar energy across the new land granted to the UW Farm. The land is between Foege, Hitchcock, and the Ocean Sciences building on South campus.

Our goal was to give a visual graph of how much sun hit different parts of the property so that the farm could decide where the best areas to plant would be. This plot could be overlayed onto a map of the property to tell where the best growing conditions would be.

We decided to model the land as a two rectangles in the xy-plane, with building walls of varying heights in the z-direction. We plotted the area so that the buildings lined up with the xz- and yz- planes. The buildings are actually 37.5 degrees off of N-S on campus, though. Rotating our azimuth angles for the sun was much easier than plotting the buildings and field at an angle.

The illumination algorithm accepts a list of points in a coordinate plane for which to gather solar data. We also modeled the buildings surrounding the land axis-aligned rectangles. Our Horizons data source gave us solar angles for different points in time.

To calculate the illumination for one of the points, we examine the intersection of the line connecting the point and the sun and determine if that line intersects a wall or not. We reused our line-plane intersection method for casting sticks' shadows, then further refined it to determine if that intersection occurred within a certain rectangle on the plane (a "wall"). If any of the walls are between the sun and that point, the point isn't illuminated. Otherwise, the point is in the sun at the given point in time.

The illumination calculation is done (at most) once per point per wall per point in time. This is a fairly involved calculation, so we made an easy way to change the resolution over the area in question to speed things up.

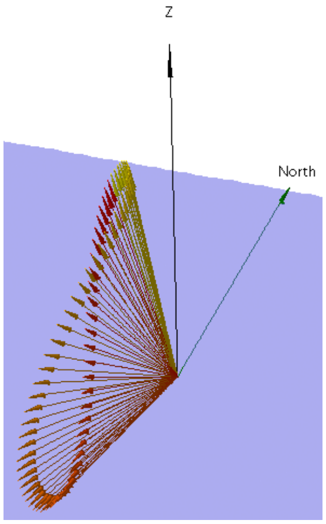
We wrote a function to get a mapping of the illumination score for a list of points, sun vectors (ray towards the sun at a point in time), and walls. The map is then plotted, giving each point a grayscale value corresponding to its illumination score in relation to the minimum and maximum scores for the set of points. Combining the illumination graph with the walls, we were able to generate plots for any set of points, walls, and sun vectors.

The UW Farm land was modeled with walls representing the buildings around the area. We used a detailed map of campus, recording pixel locations on the map to plot out where walls were, and pictures to estimate the height of the buildings. Lists of points in the two main areas that the farm is considering for planting were generated using our helper function. We got solar data during summer and winter solstice as well as the vernal equinox, and drew the illumination graph in Sage. See the resulting plots below.

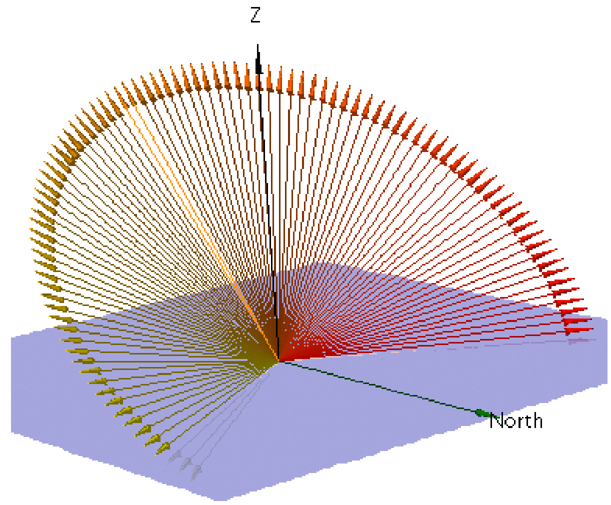
At the solstices, there was a well lit area just off of the middle of Foege that we hadn't predicted. During equinox, there is a well lit triangle on the north end of the field that would be good for planting.

The UW Farm could use our model to help them plan where to plant. This is a very real and neat application of our program.

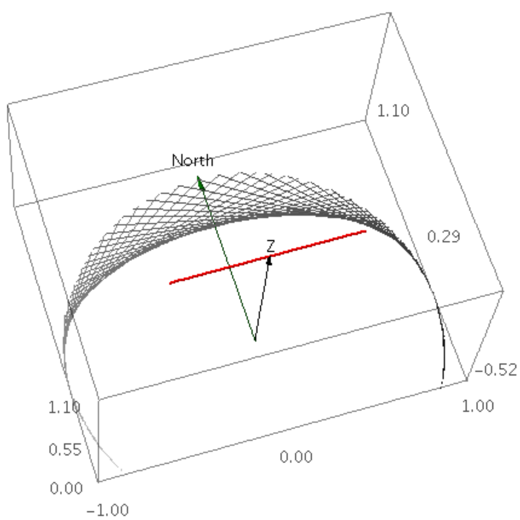
Images



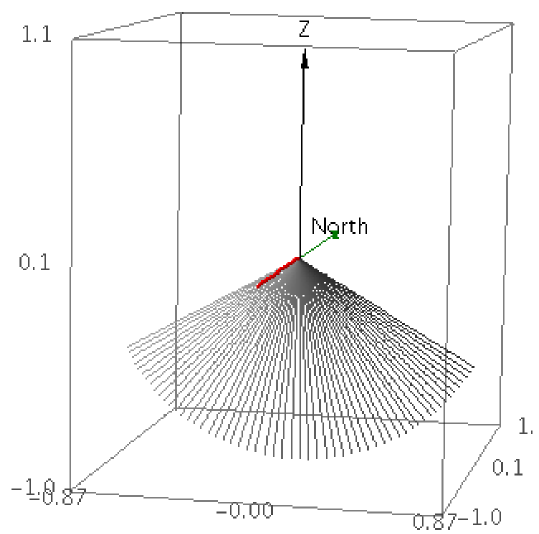
1. Sun at noon every 2 days for 1 year starting May 25, 2010



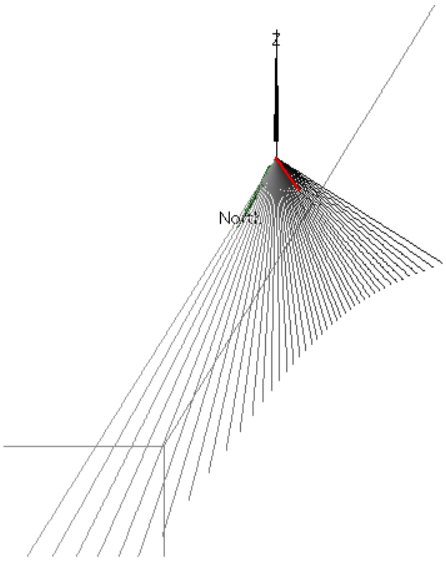
2. Sun every 10 minutes on May 25th, 2010 (used for plots below)



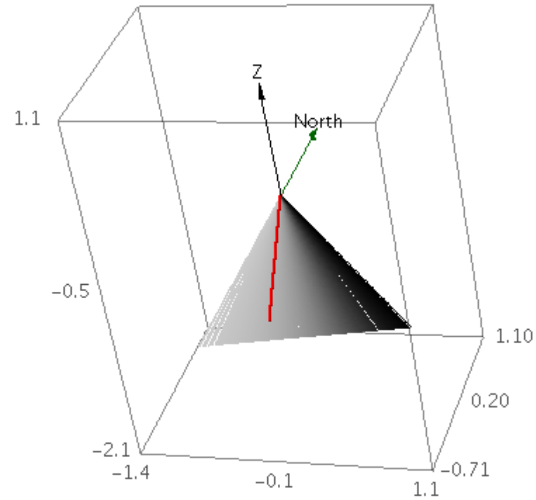
3. Stick (red) raised to a height above the ground and its shadow projected throughout a whole day.



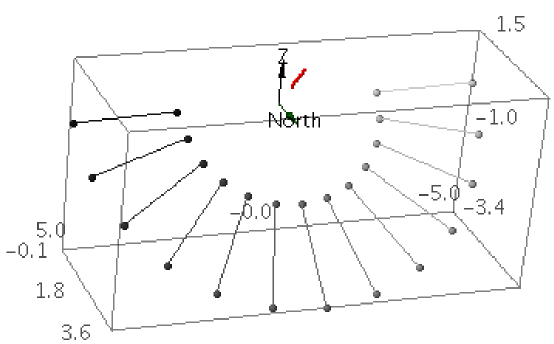
4. Stick attached to wall aligned with north-south line



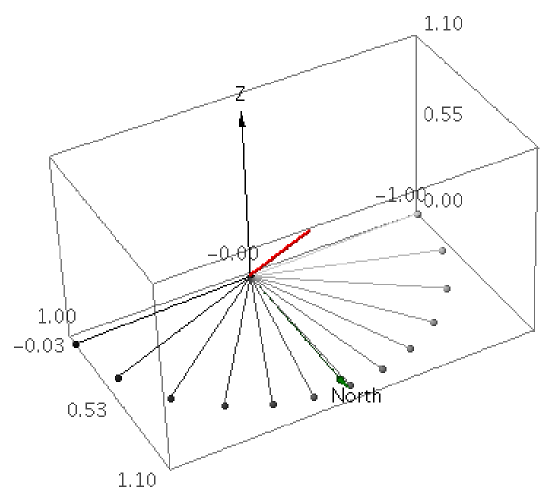
5. Model of the sundial on the side of the Physics & Astronomy building during a day in late Spring



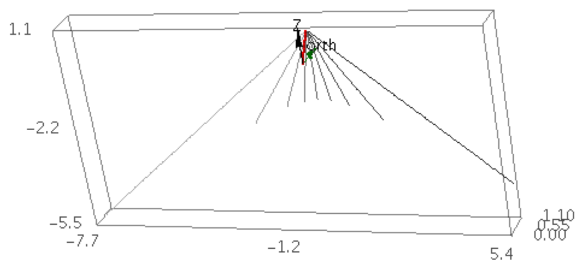
6. Physics & Astronomy sundial for a full day at equinox (notice that the shadow forms a straight line)



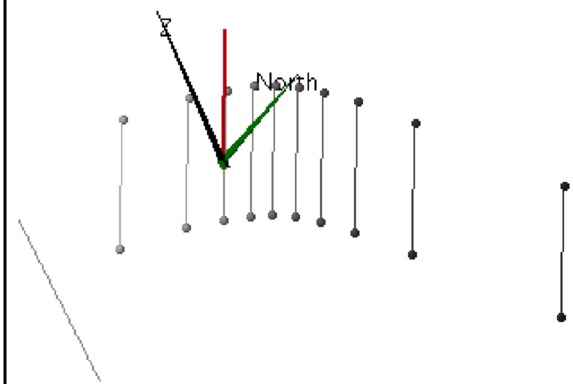
7. Equatorial sundial (stick angle = latitude)



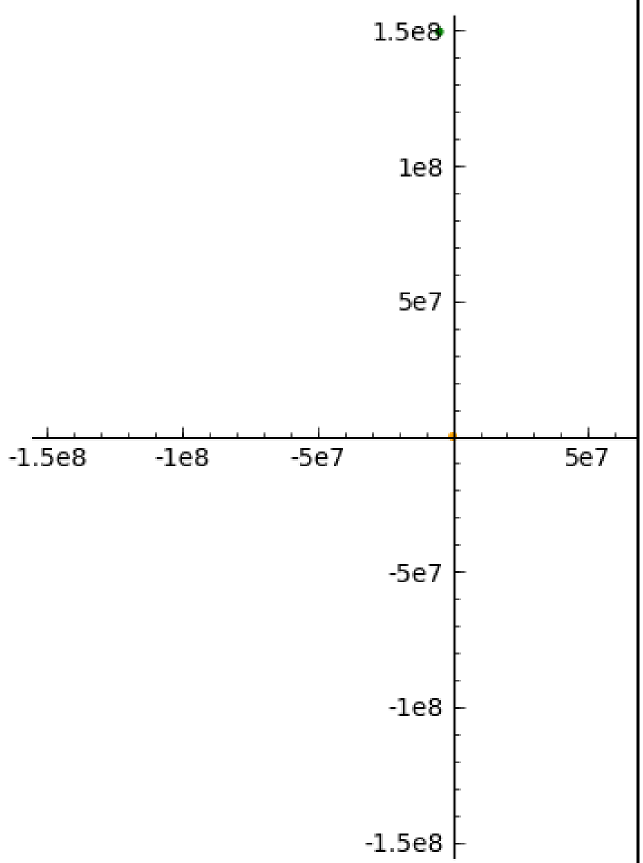
8. Horizontal sundial



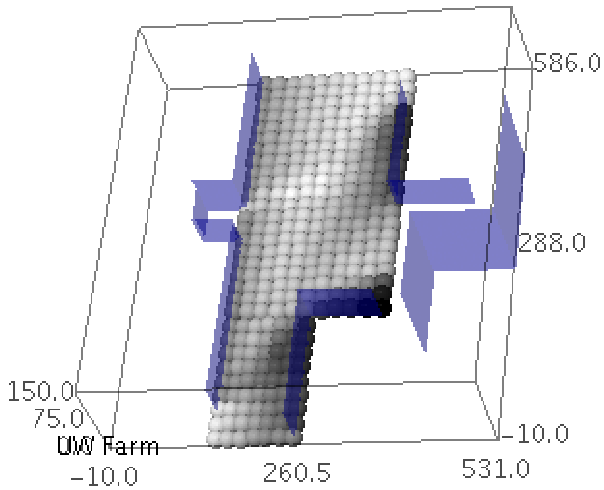
9. Vertical sundial



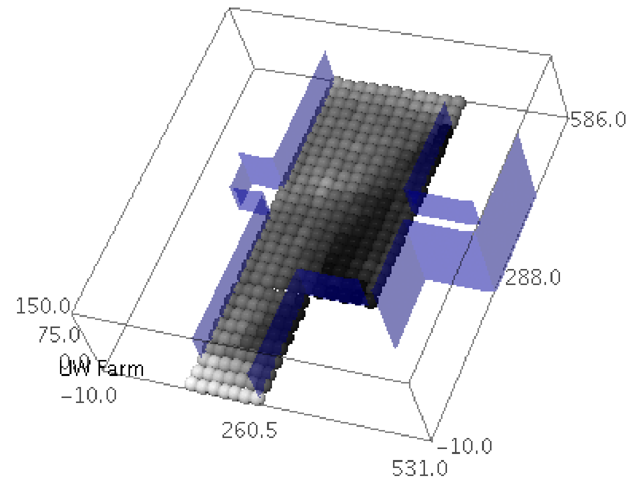
10. Polar sundial (lines all parallel)



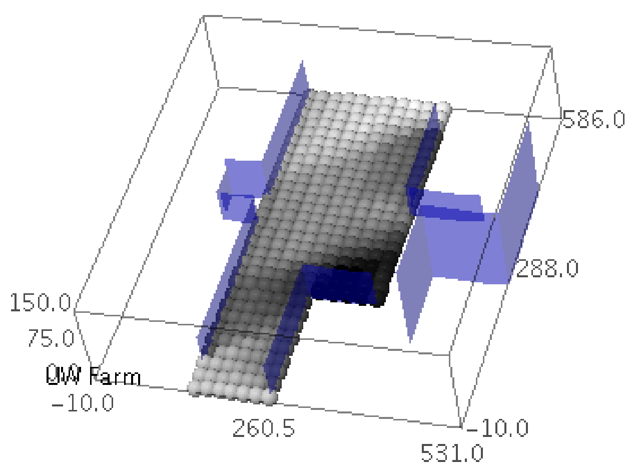
11. Earth 25% of the way through the year (time-wise). Notice the Earth is more than 25% through its orbit (top)



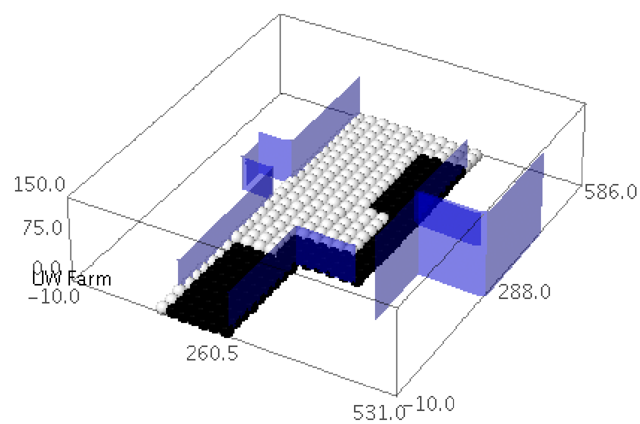
12. Illumination plot – summer solstice



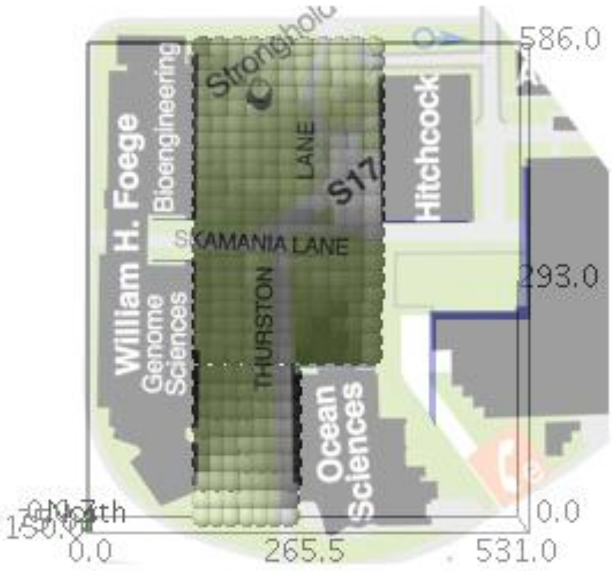
13. Illumination plot – winter solstice



14. Illumination plot – spring equinox



15. Illumination plot – single sun vector



16. Illumination plot – overlay with campus map

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