Exercise Set 1:
Prime Numbers

Math 414, Winter 2010, University of Washington

Due Wednesday, January 13, 2010

1. Compute $\gcd(2010, 1235)$ by hand.

2. Use the prime sieve describe in the book to find all primes up to 100.

3. Prove that there are infinitely many primes of the form $6x - 1$.

4. (a) So far 47 Mersenne primes $2^p - 1$ have been discovered. Give a guess, backed up by an argument, about when the next Mersenne prime might be discovered. You will have to do some online research to find the dates when Mersenne primes have been discovered in the past.

   (b) EFF will award $150,000 to the first individual or group who discovers a prime with at least 100 million digits (it will very likely be a Mersenne prime). Based on your answer to the first part of this problem, do you think you are likely to see the discovery of a 100 million digit Mersenne prime?

5. Let a, b, c, n be integers. Prove that

   (a) if $a \mid n$ and $b \mid n$ with $\gcd(a, b) = 1$, then $ab \mid n$.

   (b) if $a \mid bc$ and $\gcd(a, b) = 1$, then $a \mid c$.

6. Let a, b, c, d, and m be integers. Prove that

   (a) if $a \mid b$ and $b \mid c$ then $a \mid c$.

   (b) if $a \mid b$ and $c \mid d$ then $ac \mid bd$.

   (c) if $m \neq 0$, then $a \mid b$ if and only if $ma \mid mb$.

   (d) if $d \mid a$ and $a \neq 0$, then $|d| \leq |a|$.

7. (Do this by hand.) Compute the greatest common divisor of $a = 323$ and $b = 437$ using the algorithm described in class that involves quotients and remainders (i.e., do not just factor a and b). Show your work.

8. Roughly how long does it take Sage to compute the greatest common divisor of two random 100,000 digit numbers?

9. Suppose that $3 = (a + b\sqrt{-5})(c + d\sqrt{-5})$ with $a, b, c, d \in \mathbf{Z}$. Prove that one of $a + b\sqrt{-5}$ or $c + d\sqrt{-5}$ is ± 1.