

To print higher-resolution math symbols, click the  
**Hi-Res Fonts for Printing** button on the jsMath control panel.

[Edit this.](#) [Download.](#) [Other published documents...](#)

# Project 2 [Math 583e]: Combinatorics of Abelian Varieties

4 minutes ago by admin

## Combinatorics of Abelian Varieties

**Project 2: Math 583e [Spring 2009]**

**Sourav Sen Gupta**

### Goal of the Project

The idea of this project originated from an email from William Stein (to the class) proposing a few project possibilities:

"Compute some sort of multigraph or other combinatorial structure associated to intersections of all subsets of  $\{A_f : f \text{ is a newform of level dividing } N\}$ . Do this for  $N < 100$ , say. There is code in Sage already to compute the intersections, so just need to run it, store the results, decide what they are combinatorially, make tables/pictures." - William Stein

To start off with, this was the goal of this project. Later, it got a bit modified to include the oldforms along with the newforms at any level  $N$ . So, the goal of this project can be stated as follows:

*Compute some sort of multigraph or other combinatorial structure associated to intersections of all subsets of  $\{A_f : f \text{ is a newform or oldform of level dividing } N\}$ , make pictures and/or store the results in an accessible format.*

## Outcome of the Project

1. Computed the graph structures associated with the intersections of all subsets of  $\{A_f : f \text{ is a newform or oldform of level dividing } N\}$  for  $N \leq 500$  [using sagemath]
2. Pickled (.soj) the data in chunks of 100 levels in each file at my home directory [/home/souravsg](#) [on sagemath]
3. Tried to produce a code to parse the data and make it into a database (sage.databases.db) but failed [the .py is at [/home/souravsg](#)]
4. Made a separate handler of the data which works by (crudely) loading the .soj files and accessing the data thereof [included here]

## Background Material

Let  $f = \sum_{n=1}^{\infty} a_n q^n \in S_2(\Gamma_0(N))$  be a newform on  $\Gamma_0(N)$ . Here  $q = e^{2\pi iz}$  is a holomorphic function on the upper half plane  $\mathbb{H}$  such that  $f(z)dz$  remains invariant under the action of  $\Gamma_0(N)$ . Also,  $f$  is an eigenvector for all the Hecke operators  $T_n$ , and the relation is  $T_n(f) = a_n f$ . The ring  $\mathbb{T} = \mathbb{Z}[\dots, T_n, \dots]$  generated by these Hecke operators is a commutative ring that is finite as a  $\mathbb{Z}$ -module. This  $\mathbb{T}$  is called the Hecke algebra.

We can associate an ideal  $I_f \in \mathbb{T}$  to the newform  $f$  in the following fashion

$$I_f = \text{Ann}_{\mathbb{T}}(f) = \{t \in \mathbb{T} : tf = 0\}$$

. If  $\mathcal{O}_f = \mathbb{Z}[\dots, a_n, \dots]$  be the ring generated by the Fourier coefficients  $a_n$  of  $f$ , then we obtain a short exact sequence  $0 \rightarrow I_f \rightarrow \mathbb{T} \rightarrow \mathcal{O}_f \rightarrow 0$ .

If we define the extended upper half plane as  $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup i\infty$ , then one can also view  $f(z)dz$  to be a holomorphic differential on

$$X_0(N)(\mathbb{C}) = \Gamma_0(N) \backslash \mathbb{H}^*$$

where  $X_0(N)$  is a modular curve, i.e, an algebraic curve of genus  $g = \dim S_2(\Gamma_0(N))$  with a canonical representation over  $\mathbb{Q}$ .

The Hecke algebra  $\mathbb{T}$  acts on the group  $\text{Div}(X_0(N))$ , which in turn induces an action of  $\mathbb{T}$  on the Jacobian  $J_0(N) = \text{Jac}(X_0(N))$ . This Jacobian  $J_0(N)$  is a projective algebraic variety of dimension  $g$  defined over  $\mathbb{Q}$  which has an algebraic group structure. Hence,  $J_0(N)$  is an algebraic variety.

Now we can associate a new algebraic structure with our newform  $f$ . We know that  $J_0(N)$  is an abelian variety and so is  $I_f J_0(N)$ . Thus the construction

$$A_f = J_0(N)/I_f J_0(N)$$

is an abelian variety associated with  $f$ . If we consider  $K_f = \text{Frac}(\mathcal{O}_f) = \mathbb{Q}(\dots, a_n, \dots)$ , then  $A_f$  is an abelian variety of dimension  $[K_f : \mathbb{Q}]$  and  $\mathcal{O}_f \subset \text{End}(A_f/\mathbb{Q})$ . The action of  $\mathbb{T}$  on  $J_0(N)$  induces an action of  $\mathbb{T}/I_f$  on  $A_f$  as  $I_f$  acts on  $A_f$  as 0.

These abelian varieties  $A_f$  serve the role of primes in the study of modular abelian varieties and any modular abelian variety is a product of these  $A_f$ 's. If we consider any  $J_0(N)$ , then it is isomorphic to a product of  $A_f$ 's such that the level of  $f$  divides  $N$ . For an example

$$J_0(N) \sim A_{f_1} \times A_{f_2} \times A_{f_3} \times A_{f_6} \times A_{f_{20}}$$

where  $f_i$  is a newform of level 389 and dimension of each simple abelian subvariety  $A_{f_i}$  is  $i$ . We can compute this as follows:

```
J = J0(389)
print(J)
D = J.decomposition()
print(D)
```

```
Abelian variety J0(389) of dimension 32
[
Simple abelian subvariety 389a(1,389) of dimension 1 of J0(389),
Simple abelian subvariety 389b(1,389) of dimension 2 of J0(389),
Simple abelian subvariety 389c(1,389) of dimension 3 of J0(389),
Simple abelian subvariety 389d(1,389) of dimension 6 of J0(389),
Simple abelian subvariety 389e(1,389) of dimension 20 of J0(389)
]
```

Now, if we consider the intersection of a pair of  $A_f$ 's, then we obtain a group. For example, taking the intersection of  $A_{f_1}$  and  $A_{f_2}$  as in the above decomposition will give us a finite group with invariants  $[2, 2]$  over  $\mathbb{Q}$  which is a subgroup of  $A_{f_1}$ . The intersection also produces a simple abelian subvariety of dimension 0, illustrating the fact that  $A_{f_1}$  and  $A_{f_2}$  are simple.

```
A_f_1 = D[0]
```

```
A_f_2 = D[1]
(G,A) = A_f_1.intersection(A_f_2)
print(G)
print(A)
```

Finite subgroup with invariants [2, 2] over QQ of Simple abelian subvariety 389a(1,389) of dimension 1 of J0(389)  
Simple abelian subvariety of dimension 0 of J0(389)

That's fine. But, what does it tell us? Is there anything significant about this group sitting at the intersection of the two abelian varieties? It turns out that this intersection corresponds to congruences between the associated eigenforms  $f_1$  and  $f_2$ . In general, if  $f$  and  $g$  are two newforms which are not galois conjugates, then the intersection  $A_f \cap A_g$  is non-empty and the integer  $|A_f \cap A_g|$  is of significance. This cardinality relates  $f$  and  $g$  through some congruence relation, but the relation is not completely understood.

It has been proven that if a prime  $p$  divides this cardinality, then there exists a prime  $\wp \subset K_f K_g$  with residue characteristic  $p$  such that

$$a_q(f) \equiv a_q(g) \pmod{\wp} \quad \forall q \nmid N$$

But the converse of this statement is not always true (though it quite frequently is). For a counterexample, one may consider the case  $N = 431$ .

## Computation for the Project

For this project, our goal is to compute all such intersections between pairs of abelian subvarieties at every level  $N \leq 500$ . Thereafter, we will make the data into a graph-like structure with the varieties  $A_f$  at the nodes. Two nodes  $A_f$  and  $A_g$  will be connected by an edge if and only if the intersection  $A_f \cap A_g$  is non-empty. In such a case, we will label the edge by the group obtained from the intersection. In case of a specific level 389, say, the code to generate the graph is as follows:

```
N = 389
J = J0(N)
D = J.decomposition()
S = Subsets(D,2)
GraphN = Graph({})
```

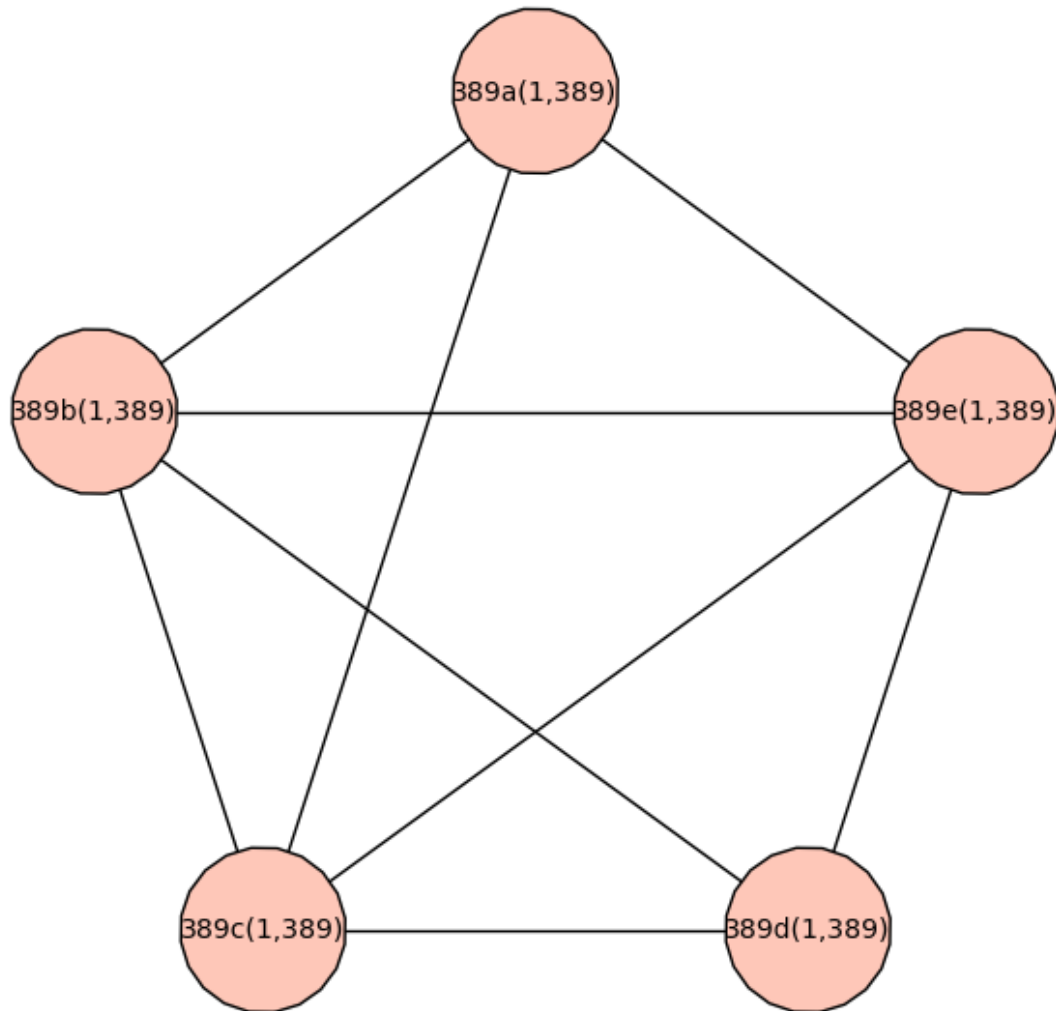
```

for A in S:
    GroupN,VarietyN = A[0].intersection(A[1])

    if GroupN.order() > 1:
        GraphN.add_edge(A[0].label(),A[1].label(),GroupN)

GraphN.plot(layout='circular', vertex_size=3000)

```



As one can notice, the above graph is constructed by putting the abelian varieties on the vertices and the edges denote congruence relations between the associated eigenforms. I have computed these graphs for all the levels  $N \leq 500$  and have stored the data in pickled .soj file format.

## Accessing the Data

The data is split over 5 files, each containing 100 levels, and the files can be obtained at </home/souravsg> on the sagemath server. The naive way to access the data is illustrated in the following steps:

1. Load the data file using the correct path

```
GraphList = load('/home/souravsg/data100.sobj')
```

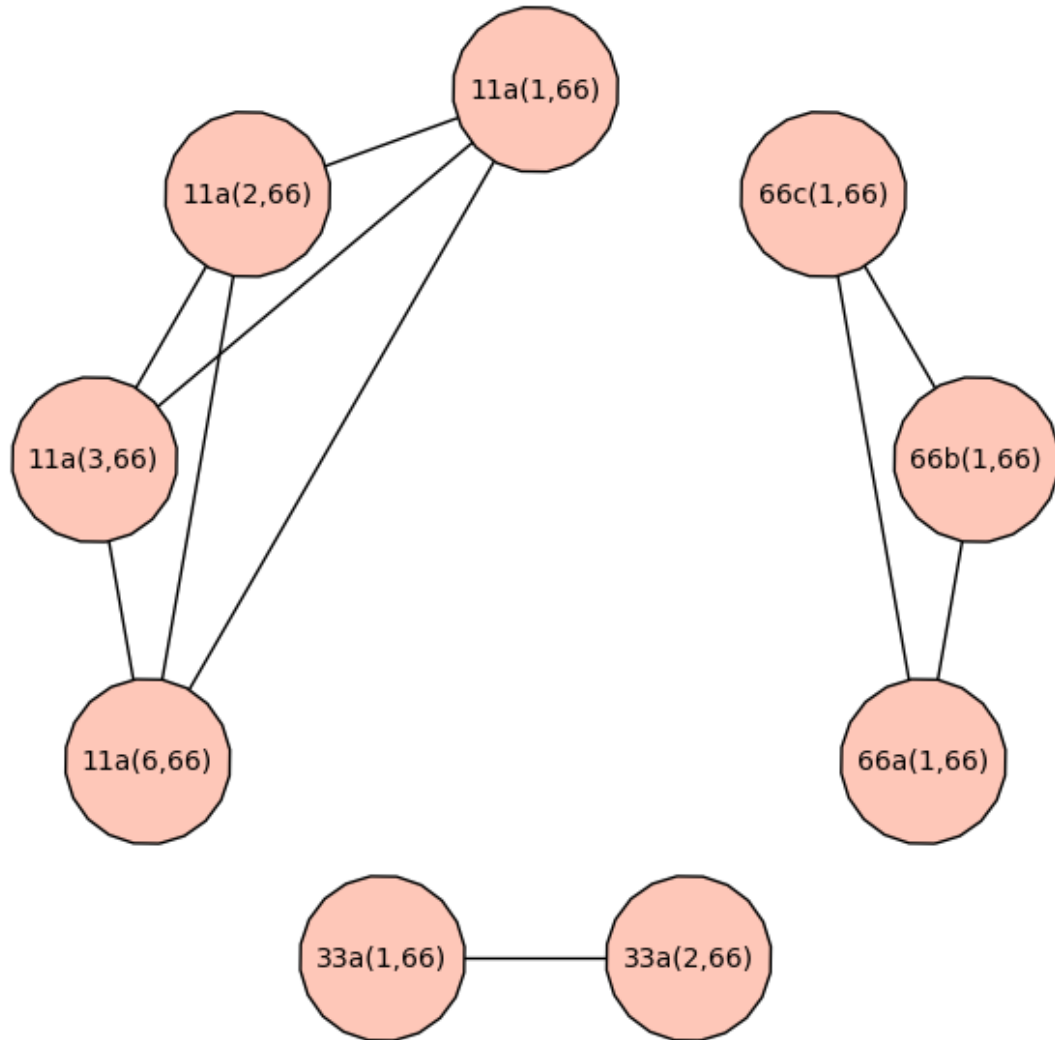
2. The data `GraphList` is a list of lists, the elements being of the format `[M, GraphM]` for each level  $M \leq 100$  in this case. So, to access the graph for level  $M$ , fetch `GraphList[M-1]` (list indexed from 0)

```
M = 66
(M, GraphM) = GraphList[M-1]
```

3. Now, one can either plot `GraphM` or obtain different information from it

```
print 'Level:',M
print 'The abelian varieties:',GraphM.vertices()
GraphM.plot(layout='circular', vertex_size=3000)
```

```
Level: 66
The abelian varieties: ['11a(1,66)', '11a(2,66)', '11a(3,66)',
'11a(6,66)', '33a(1,66)', '33a(2,66)', '66a(1,66)', '66b(1,66)',
'66c(1,66)']
```



To obtain intersection information between specific abelian varieties, one can do the following:

```
A_1 = '11a(3,66)'
A_2 = '11a(6,66)'
print 'Is A_1 intersect A_2 non-empty:', GraphM.has_edge(A_1, A_2)
G = GraphM.edge_label(A_1, A_2)
print 'The intersection:', G[0]
print 'Cardinality of the intersection:', G[0].order()
```

```
Is A_1 intersect A_2 non-empty: True
The intersection: Finite subgroup with invariants [5] over QQ of
Simple abelian subvariety 11a(3,66) of dimension 1 of J0(66)
Cardinality of the intersection: 5
```

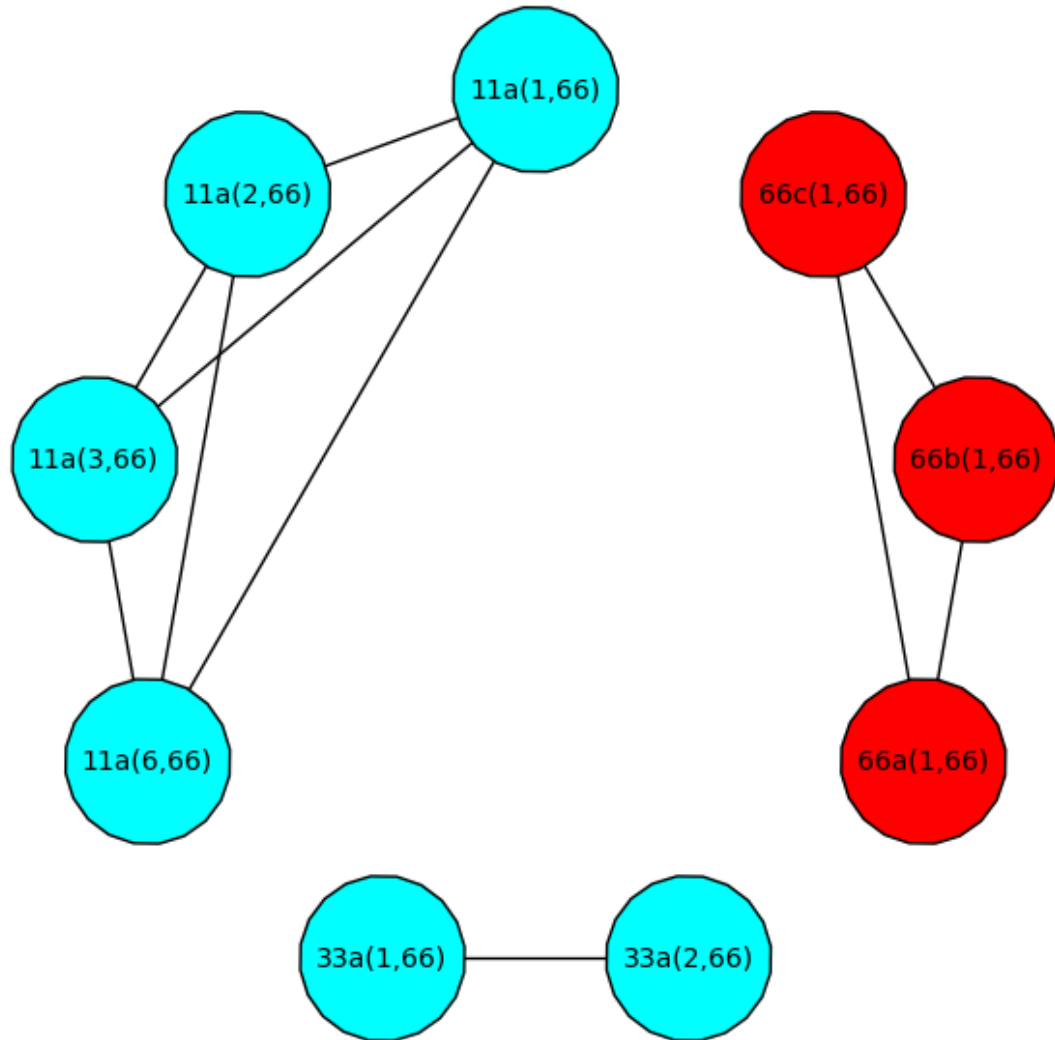
```
A_3 = '33a(1,66)'
```

```
A_4 = '66b(1,66)'  
print 'Is A_3 intersect A_4 non-empty:', GraphM.has_edge(A_3, A_4)  
Is A_3 intersect A_4 non-empty: False
```

In the previous examples, we have seen that  $M = 389$  had all 5 newforms on the graph whereas  $M = 66$  had both new and old forms. Also, notice that there are no intersections (edges) between the oldforms and the newforms in the case when  $M = 66$ , as depicted in the graph above. But, sometimes we are more interested to know if there are intersections between old and newforms at any level. One can easily check this visually by a modified graph plotting function as follows:

```
M = 66  
(M, GraphM) = GraphList[M-1]  
  
NewformsM = J0(M).new_subvariety().decomposition()  
NewnodesM = [Newf.label() for Newf in NewformsM]  
OldformsM = J0(M).old_subvariety().decomposition()  
OldnodesM = [Oldf.label() for Oldf in OldformsM]  
P = [NewnodesM, OldnodesM]  
  
GraphM.plot(layout='circular', vertex_size=3000, partition=P)
```



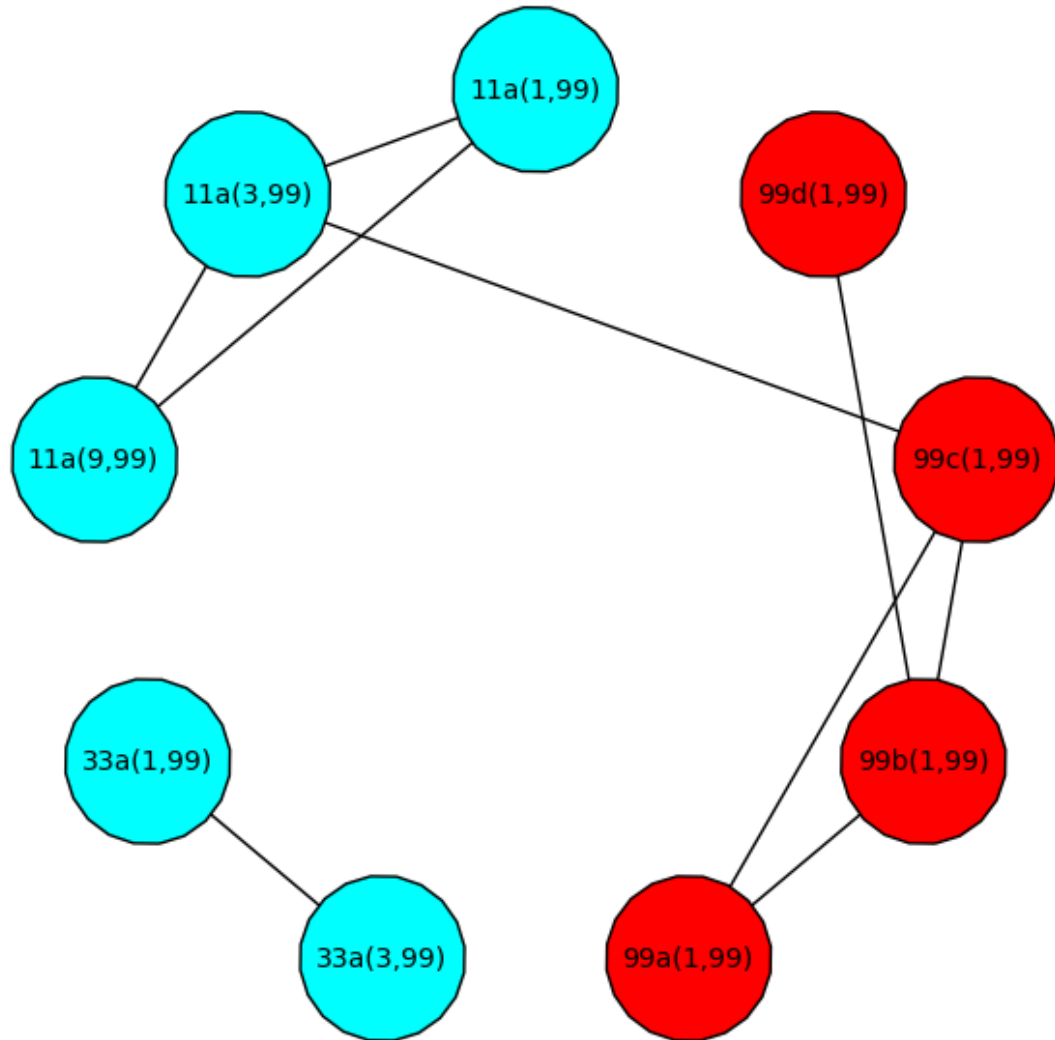


This clearly and visually shows that there are no such interesting intersections in the case  $M = 66$ . But let us take a quick look at the level  $M = 99$  to see such a case taking place.

```
M = 99
(M, GraphM) = GraphList[M-1]

NewformsM = J0(M).new_subvariety().decomposition()
NewnodesM = [Newf.label() for Newf in NewformsM]
OldformsM = J0(M).old_subvariety().decomposition()
OldnodesM = [Oldf.label() for Oldf in OldformsM]
P = [NewnodesM, OldnodesM]

GraphM.plot(layout='circular', vertex_size=3000, partition=P)
```



```

A_old = '11a(3,99)'
A_new = '99c(1,99)'
G = GraphM.edge_label(A_old,A_new)
print 'Intersection:',G[0]
print 'Cardinality',G[0].order()

```

Intersection: Finite subgroup with invariants [3, 3] over  $\mathbb{Q}$  of Simple abelian subvariety 99c(1,99) of dimension 1 of  $J_0(99)$   
Cardinality 9

## Part that did not work

I was also looking through the database and all the graphs to find any significant combinatorial property or structure of these graphs. If there exists any structure,

then some useful facts regarding the congruence of eigenforms can be predicted based on those observations. But I could not find anything specifically interesting in my search. Maybe there exists an underlying combinatorial model, but it was just out of my reach.

I have also tried to put all this data in terms of a sage database. But I guess I am missing some important understanding of the sage database structure and hence could not convert the .sobj data into sage.databases.db format. I have posted my try (in vein) as a .py file at [/home/souravsg](#) on the sagemath server so that someone who is interested can fix it and get it to work.

## Acknowledgements

I would like to thank William Stein for offering such an interesting entry level course towards computing with modular forms. It was a great learning experience throughout. Thanks to William once again for offering this project and showing me the basic Sage commands I could use for accomplishing my goal.

I will also like to thank Tom Boothby for providing me a sagemath account and helping me with sagemath server issues and most certainly for patiently tolerating my stupid questions about Sage and the sagemath server operations.

## References

1. Course notes for Math 583e: "Computing Conjectural Orders of Shafarevich-Tate Groups of Modular Abelian Varieties", William Stein, University of Washington, 2009
2. Lecture notes for Math 252: "Modular Abelian Varieties", William Stein, Harvard University, 2003
3. "Lectures on Serre's Conjectures", Kenneth Ribet and William Stein, IAS/Park City Mathematics Series, AMS, 2001
4. Numerous Sage documentations and help files

