Math 480 (Spring 2007): Homework 6

Due: Monday, May 7

There are 5 problems. Each problem is worth 6 points and parts of multipart problems are worth equal amounts. You may work with other people and use a computer, unless otherwise stated. Acknowledge those who help you.

1. Calculate the following Legendre symbols by hand (you may use the quadratic reciprocity law): $\left(\frac{4}{10007}\right)$, $\left(\frac{3}{37}\right)$, $\left(\frac{5}{11}\right)$, $\left(\frac{5!}{7}\right)$.

2. Let G be an abelian group and let n be a positive integer.

   (a) Prove that the map $\varphi : G \to G$ given by $\varphi(x) = x^n$ is a group homomorphism.

   (b) Prove that the subset H of G of squares of elements of G is a subgroup.

3. Give an example of an abelian group G and two distinct subgroups H and K both of index 2. Note that G will not be cyclic.

4. (*) Prove that for any $n \in \mathbb{Z}$ the integer $n^2 + n + 1$ does not have any divisors of the form $6k - 1$. (Hint: First reduce to the case that $6k - 1$ is prime, by using that if p and q are primes not of the form $6k - 1$, then neither is their product. If $p = 6k - 1$ divides $n^2 + n + 1$, it divides $4n^2 + 4n + 4 = (2n + 1)^2 + 3$, so -3 is a quadratic residue modulo p. Now use quadratic reciprocity to show that -3 is not a quadratic residue modulo p.)

5. For each of the following equations, either find all integer solutions with $0 \leq x < p$ or prove that no solutions exist:

   (a) $x^2 + 2x + 3 \equiv 0 \pmod 7$, where $p = 7$.

   (b) $x^2 - x + 7 \equiv 0 \pmod{11}$, where $p = 11$.

   (c) $x^2 + x + 1 \equiv 0 \pmod 2$, where $p = 2$.

   (d) $x^2 - 3 \equiv 0 \pmod{389}$, where $p = 389$.

   (e) $x^2 + x + 1 \equiv 0 \pmod 3$, where $p = 3$.

   (f) $2x^2 + 3x - 2 \equiv 0 \pmod 5$, where $p = 5$.