Math 480 (Spring 2007): Homework 1

Due: Monday, April 2

There are 7 problems. Each problem is worth 5 points, and parts of multi-part problems are worth equal amounts.
Office Hours. My official office hours are on Thursdays 4–6pm in Padelford C423.

1. (This problem must be done without help from anyone else.) Let a, b, c, d, and m be integers. Prove that
   (a) if $a \mid b$ and $b \mid c$ then $a \mid c$,
   (b) if $a \mid b$ and $c \mid d$ then $ac \mid bd$,
   (c) if $m \neq 0$, then $a \mid b$ if and only if $ma \mid mb$, and
   (d) if $d \mid a$ and $a \neq 0$, then $|d| \leq |a|$.

2. (This problem must be done by hand without help from anyone else.) In each of the following, apply the division algorithm to find q and r such that $a = bq + r$ and $0 \leq r < |b|$:
$$a = 300, b = 17, \quad a = 729, b = 31, \quad a = 300, b = -17, \quad a = 389, b = 4.$$

3. (a) (Do this part by hand.) Compute the greatest common divisor of 323 and 437 using the algorithm described in class that involves quotients and remainders (i.e., do not just factor a and b).
   (b) Compute by any means the greatest common divisor
$$\gcd(31415926535897932384626433\,8, 27182818284590452353602874\,7).$$

4. (a) Suppose a, b and n are positive integers. Prove that if $a^n \mid b^n$, then $a \mid b$.
   (b) Suppose p is a prime and a and k are positive integers. Prove that if $p \mid a^k$, then $p^k \mid a^k$.

5. (a) Prove that if a positive integer n is a perfect square, then n cannot be written in the form $4k + 3$ for k an integer. (Hint: Compute the remainder upon division by 4 of each of $(4m)^2$, $(4m + 1)^2$, $(4m + 2)^2$, and $(4m + 3)^2$.)
   (b) Prove that no integer in the sequence
$$11, 111, 1111, 11111, 111111, \ldots$$
   is a perfect square. (Hint: $111 \cdots 111 = 111 \cdots 108 + 3 = 4k + 3$.)

6. Prove that a positive integer n is prime if and only if n is not divisible by any prime p with $1 < p \leq \sqrt{n}$.

7. So far 44 Mersenne primes $2^p - 1$ have been discovered. Give a guess, backed up by an argument, about when the next Mersenne prime might be discovered (you will have to do some online research).