Chapter 8

Modular Symbols of Arbitrary Weight and Level

In this chapter we explain how to generalize the notion of modular symbols given in Chapter 3 to higher weight and more general level.

Modular symbols are a formalism that make it fairly easy and elementary to compute with homology or cohomology related to certain Kuga-Sato varieties (these are $\mathcal{E} \times_X \cdots \times_X \mathcal{E}$, where X is a modular curve and $\mathcal{E}$ is the univeral elliptic curve over it). It is not necessary to know anything about these Kuga-Sato varieties in order to compute with modular symbols.

This chapter is about spaces of modular symbols and how to compute with them. It is by far the most important chapter in this book. The algorithms that build on the theory in this chapter are central to all the computations we will do later in the book.

This chapter follows Loïc Merel's paper [Mer94]. First we define modular symbols of weight $k \geq 2$. Then we define the corresponding Manin symbols, and state a theorem of Merel-Shokurov, which gives all relations between Manin symbols. (The proof of the Merel-Shokurov theorem is beyond the scope of this book.) Next we describe how the Hecke operators act on both modular and Manin symbols, and how to compute trace and inclusion maps between spaces of modular symbols of different levels. We close the chapter with a discussion of computations with modular symbols over finite fields.

Not only are modular symbols useful for computation, but they have been used to prove theoretical results about modular forms. For example, certain technical calculations with modular symbols are used in Loic Merel's proof of the uniform boundedness conjecture for torsion points on elliptic curves over number fields; modular symbols arise, e.g., in order to understand linear independence of Hecke operators. Another example is Grigor Grigorov's in-progress Ph.D. thesis, which distills hypotheses about Kato's Euler system in K_2 of modular curves to a simple formula involving modular symbols (when the hypotheses are satisfied, one obtains a lower bound on the Shafarevich-Tate group of an elliptic curve).

8.1 Modular Symbols

We recall from Chapter 3 the free abelian group $\mathbb{M}$ of modular symbols. We view these as elements of the relative homology of the extended upper half plane $\mathfrak{h}^* = \mathfrak{h} \cup \mathbb{P}^1(\mathbb{Q})$ relative to the cusps. The group $\mathbb{M}$ is the free abelian group on symbols $\{\alpha, \beta\}$ with

$$\alpha, \beta \in \mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$$

subject to the relations

$$\{\alpha, \beta\} + \{\beta, \gamma\} + \{\gamma, \alpha\} = 0,$$

for all $\alpha, \beta, \gamma \in \mathbb{P}^1(\mathbb{Q})$. More precisely, $\mathbb{M} = (F/R)/(F/R)_{\mathrm{tor}}$, where F is the free abelian group on all pairs (α, β) and R is the subgroup generated by all elements of the form $(\alpha, \beta) + (\beta, \gamma) + (\gamma, \alpha)$. Note that $\mathbb{M}$ is a huge free abelian group of countable rank.

Now fix an integer $k \geq 2$. Let $\mathbb{Z}_{k-2}[X, Y]$ be the abelian group of homogeneous polynomials of degree $k-2$ in two variables X, Y (so $\mathbb{Z}_{k-2}[X, Y]$ is isomorphic to $\mathrm{Sym}^{k-2}(\mathbb{Z})$ as a group, but certain natural actions are different). Set

$$\mathbb{M}_k = \mathbb{Z}_{k-2}[X, Y] \otimes_{\mathbb{Z}} \mathbb{M},$$

which is a torsion-free abelian group whose elements are sums of expressions of the form $X^i Y^{k-2-i} \otimes \{\alpha, \beta\}$. For example,

$$X^3 \otimes \{0, 1/2\} - 17XY^2 \otimes \{\infty, 1/7\} \in \mathbb{M}_5.$$

Fix a finite index subgroup G of $\mathrm{SL}_2(\mathbb{Z})$. Define a *left action of* G on $\mathbb{Z}_{k-2}[X, Y]$ as follows. If $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$ and $P(X, Y) \in \mathbb{Z}_{k-2}[X, Y]$, let

$$(gP)(X, Y) = P(dX - bY, -cX + aY).$$

Note that if we think of $z = (X, Y)$ as a column vector, then

$$(gP)(z) = P(g^{-1}z),$$

since $g^{-1} = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$. The reason for the inverse is so that this is a left action instead of a right action, e.g., if $g, h \in G$, then

$$((gh)P)(z) = P((gh)^{-1}z) = P(h^{-1}g^{-1}z) = (hP)(g^{-1}z) = (g(hP))(z).$$

Let G act on the left on $\mathbb{M}$ by

$$g\{\alpha, \beta\} = \{g(\alpha), g(\beta)\}.$$

Here G acts via linear fractional transformations, so if $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then

$$g(\alpha) = \frac{a\alpha + b}{c\alpha + d}.$$

For example, useful special cases to remember are that if $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ then

$$g(0) = \frac{b}{d} \qquad \text{and} \qquad g(\infty) = \frac{a}{c}.$$

We now combine these two actions to obtain a left action of G on $\mathbb{M}_{k-2}$, which is given by

$$g(P \otimes \{\alpha, \beta\}) = (gP) \otimes \{g(\alpha), g(\beta)\}.$$

For example,

$$\begin{aligned} \begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix} .(X^3 \otimes \{0, 1/2\}) &= (-3X - 2Y)^3 \otimes \left\{-\frac{2}{3}, -\frac{5}{8}\right\} \\ &= (-27X^3 - 54X^2Y - 36XY^2 - 8Y^3) \otimes \left\{-\frac{2}{3}, -\frac{5}{8}\right\}. \end{aligned}$$

We will often write $P(X, Y)\{\alpha, \beta\}$ for $P(X, Y) \otimes \{\alpha, \beta\}$.

Definition 8.1.1 (Modular Symbols)**.** Let $k \geq 2$ be an integer and let G be a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$. The space $\mathbb{M}_k(G)$ of weight k modular symbols for G is the quotient of $\mathbb{M}_k$ by all relations $gx - x$ for $x \in \mathbb{M}_k$ and by any torsion.

Note that $\mathbb{M}_k$ is a torsion free abelian group, and it is a nontrivial fact that $\mathbb{M}_k$ has finite rank. We denote modular symbols for G in exactly the same way we denote elements of $\mathbb{M}_k$, but with surrounding text that hopefully makes the group G clear.

The space of *modular symbols over a ring R* is

$$\mathbb{M}_k(G, R) = \mathbb{M}_k(G) \otimes_{\mathbb{Z}} R.$$

8.2 Manin Symbols

Let G be a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and $k \geq 2$ an integer. Just as in Chapter 3 it is possible to compute $\mathbb{M}_k(G)$ using a computer, despite that, as defined above, $\mathbb{M}_k(G)$ is the quotient of one infinitely generated abelian group by another one. This section is about Manin symbols, which are a distinguished subset of $\mathbb{M}_k(G)$ that lead to a finite presentation for $\mathbb{M}_k(G)$. Formulas written in terms of Manin symbols are frequently much easier to compute using a computer than formulas in terms of modular symbols, and we will give several such formulas later in this chapter.

Suppose $P \in \mathbb{Z}_{k-2}[X, Y]$ and $g \in \mathrm{SL}_2(\mathbb{Z})$. Then the *Manin symbol* associated to this pair of elements is

$$[P, g] = g(P\{0, \infty\}) \in \mathbb{M}_k(G).$$

Notice that if $Gg = Gh$, then $[P, g] = [P, h]$, since the symbol $g(P\{0, \infty\})$ is invariant by the action of G on the left (by definition, since it is a modular symbols for G). Thus for a right coset Gg it makes sense to write $[P, Gg]$ for the symbol $[P, h]$ for any $h \in Gg$. Since G has finite index in $\mathrm{SL}_2(\mathbb{Z})$, the abelian group generated by Manin symbols is of finite rank, generated by

$$\left\{[X^{k-2-i}Y^i,\, Gg_j] \;:\; i = 0, \ldots, k-2, \quad \text{and} \quad j = 0, \ldots, r\right\},$$

where $g_0, \ldots, g_r$ run through representatives for the right cosets $G\backslash \mathrm{SL}_2(\mathbb{Z})$.

We next show that every modular symbols can be written as a $\mathbb{Z}$-linear combination of Manin symbols, so they generate $\mathbb{M}_k(G)$.

Proposition 8.2.1. *The Manin symbols generate* $\mathbb{M}_k(G)$.

Proof. The proof if very similar to that of Proposition 3.3.2 except we introduce an extra twist to deal with the polynomial part. Suppose that we are given a modular symbol $P\{\alpha, \beta\}$ and wish to represent it as a sum of Manin symbols. Because

$$P\{a/b, c/d\} = P\{a/b, 0\} + P\{0, c/d\},$$

it suffices to write $P\{0, a/b\}$ in terms of Manin symbols. Let

$$0 = \frac{p_{-2}}{q_{-2}} = \frac{0}{1},\ \frac{p_{-1}}{q_{-1}} = \frac{1}{0},\ \frac{p_0}{1} = \frac{p_0}{q_0},\ \frac{p_1}{q_1},\ \frac{p_2}{q_2}, \ldots,\ \frac{p_r}{q_r} = \frac{a}{b}$$

denote the continued fraction convergents of the rational number a/b. Then

$$p_j q_{j-1} - p_{j-1} q_j = (-1)^{j-1} \qquad \text{for } -1 \le j \le r.$$

If we let $g_j = \begin{pmatrix} (-1)^{j-1}p_j & p_{j-1} \\ (-1)^{j-1}q_j & q_{j-1} \end{pmatrix}$, then $g_j \in \mathrm{SL}_2(\mathbb{Z})$ and

$$\begin{aligned} P\{0, a/b\} &= P \sum_{j=-1}^{r} \left\{ \frac{p_{j-1}}{q_{j-1}}, \frac{p_j}{q_j} \right\} \\ &= \sum_{j=-1}^{r} g_j((g_j^{-1}P)\{0, \infty\}) \\ &= \sum_{j=-1}^{r} [g_j^{-1}P,\ g_j]. \end{aligned}$$

Since $g_j \in \mathrm{SL}_2(\mathbb{Z})$ and P has integer coefficients, the polynomial $g_j^{-1}P$ also has integer coefficients, so we introduce no denominators. □

Now that we know the Manin symbols generate $\mathbb{M}_k(G)$, we next consider the relations between Manin symbols. Fortunately, the answer is fairly simple (though the proof is not). Let

$$\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \tau = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \qquad J = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Define a *right action* of $\mathrm{SL}_2(\mathbb{Z})$ on Manin symbols as follows. If $h \in \mathrm{SL}_2(\mathbb{Z})$, let

$$[P, g]h = [h^{-1}P, gh].$$

This is a right action because $P \mapsto h^{-1}P$ is a right action, and right multiplication $g \mapsto gh$ is also a right action.

Theorem 8.2.2. *If x is a Manin symbol, then*

$$x + x\sigma = 0 \tag{8.2.1}$$

$$x + x\tau + x\tau^2 = 0 \tag{8.2.2}$$

$$x - xJ = 0. \tag{8.2.3}$$

Moreover, these are all the relations between Manin symbols, in the sense that the space $\mathbb{M}_k(G)$ of modular symbols is isomorphic to the quotient of the free abelian group on the finitely many symbols $[X^iY^{k-2-i}, Gg]$ (for $i = 0, \ldots, k-2$, and $Gg \in G\backslash \mathrm{SL}_2(\mathbb{Z})$) by the above relations and any torsion.

Proof. First we prove that the Manin symbols satisfy the above relations following Merel's proof (see [Mer94, §1.2]). Note that

$$\sigma(0) = \sigma^2(\infty) = \infty \qquad \text{and} \qquad \tau(1) = \tau^2(0) = \infty.$$

Write $x = [P, g]$, we have

$$\begin{aligned}
[P, g] + [P, g].\sigma &= [P, g] + [\sigma^{-1}.P, g\sigma] \\
&= g(P\{0, \infty\}) + g\sigma.(\sigma^{-1}.P\{0, \infty\}) \\
&= (gP)\{g(0), g(\infty)\} + (g\sigma).(\sigma^{-1}.P)\{g\sigma(0), g\sigma(\infty)\} \\
&= (gP)\{g(0), g(\infty)\} + (gP)\{g(\infty), g(0)\} \\
&= (gP)(\{g(0), g(\infty)\} + \{g(\infty), g(0)\}) \\
&= 0.
\end{aligned}$$

Also,

$$\begin{aligned}
&[P, g] + [P, g].\tau + [P, g].\tau^2 = [P, g] + [\tau^{-1}.P, g\tau] + [\tau^{-2}.P, g\tau^2] \\
&\quad = g(P\{0, \infty\}) + g\tau.(\tau^{-1}.P\{0, \infty\}) + g\tau^2.(\tau^{-2}.P\{0, \infty\}) \\
&\quad = (gP)\{g(0), g(\infty)\} + (gP)\{g\tau(0), g\tau(\infty)\}) + (gP)\{g\tau^2(0), \tau^2(\infty)\}) \\
&\quad = (gP)\{g(0), g(\infty)\} + (gP)\{g(1), g(0)\}) + (gP)\{g(\infty), g(1)\}) \\
&\quad = (gP)(\{g(0), g(\infty)\} + \{g(\infty), g(1)\} + \{g(1), g(0)\}) \\
&\quad = 0
\end{aligned}$$

Finally,

$$\begin{aligned}
[P, g] + [P, g].J &= g(P\{0, \infty\}) - gJ.(J^{-1}P\{gJ(0), gJ(\infty)\} \\
&= (gP)\{g(0), g(\infty)\} - (gP)\{g(0), g(\infty)\} \\
&= 0,
\end{aligned}$$

where we use that J acts trivially via linear fractional transformations. This proves that the listed relations are all satisifed.

That the listed relations are all relations is more difficult to prove, and we will only very briefly sketch a proof here. One approach is to show (as in [Mer94, §1.3]) that the quotient of Manin symbols by the above relations and torsion is isomorphic to a space of Šokurov symbols, which is in turn isomorphic to $\mathbb{M}_k(G)$. Merel cites [Šok80] for most of this proof. Alternatively, Merel proves (see [Mer94, Prop. 9]) that the quotient of the free complex-vector space on Manin symbols by the enumerated relations is non-canonically isomorphic (over $\mathbb{C}$) to an Eichler-Shimura cohomology group. It is a result of Eichler and Shimura (see [Shi94, Ch. 8]–Shimura's book) that their cohomology group is isomorphic to the space $M_k(G)$ of modular forms. One can show[[**Todo: Maybe I should show this, since Merel doesn't. The argument would go via breaking up into cuspidal and Eisenstein pieces and having an exact pairing between cuspidal Manin symbols and cusp forms, and noting (using Calculus) that in the relations that define $\mathbb{M}_k(G)$ we quotient out only by things that integrate to 0 with cusp forms; anything nonzero in the kernel of the map from cuspidal Manin symbols to cuspidal modular symbols would simultaneously have to pair to 0 and $\neq 0$ with some cusp form, a contradiction.**]] that $\mathbb{M}_k(G,\mathbb{R})$ is (non-canonically) isomorphic as a $\mathbb{T}$-module to $M_k(G)$, so has the same dimension. Thus the quotient of the free abelian group on the Manin symbols by the listed relations (and torsion) has the same dimension as that of $\mathbb{M}_k(G)$, which proves that the listed relations generate all relations. (Here we use that if A is a finitely generated free abelian group then rank $A = \dim_C(A \otimes_{\mathbb{Z}} \mathbb{C})$.) □

If G is a finite-index subgroup and we have an algorithm to enumerate the right cosets $G\backslash \mathrm{SL}_2(\mathbb{Z})$, and to decide which coset an arbitrary element of $\mathrm{SL}_2(\mathbb{Z})$ belongs to, then Theorem 8.2.2 and the algorithms of Chapter 7 yield an algorithm to compute $\mathbb{M}_k(G,\mathbb{Q})$. Note that if $J \in G$, then the relation $x - xJ = 0$ is automatic. Also note that the matrices σ and τ *do not commute*, so one can *not* first quotient out by the two-term σ relations, then quotient out only the remaining free generators by the τ relations, and get the right answer in general.

8.2.1 Coset Representatives and Manin Symbols

Proposition 8.2.3. *The right cosets $\Gamma_1(N)\backslash \mathrm{SL}_2(\mathbb{Z})$ are in bijection with pairs (c,d) where $c,d \in \mathbb{Z}/N\mathbb{Z}$ and $\gcd(c,d,N) = 1$. The coset containing a matrix $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ corresponds (c,d).*

Proof. This proof is copied from [Cre92, pg. 203], except in that paper Cremona works with the analogue of $\Gamma_1(N)$ in $\mathrm{PSL}_2(\mathbb{Z})$, so his result is slightly different. Suppose $\gamma_i = \left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, for $i = 1,2$. We have

$$\gamma_1\gamma_2^{-1} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} d_2 & -b_2 \\ -c_2 & a_2 \end{pmatrix} = \begin{pmatrix} a_1 d_2 - b_1 c_2 & * \\ c_1 d_2 - d_1 c_2 & a_2 d_1 - b_2 c_1 \end{pmatrix},$$

which is in $\Gamma_1(N)$ if and only if

$$c_1 d_2 - d_1 c_2 \equiv 0 \pmod{N} \tag{8.2.4}$$

and

$$a_2 d_1 - b_2 c_1 \equiv a_1 d_2 - b_1 c_2 \equiv 1 \pmod{N}. \tag{8.2.5}$$

Since the γ_i have determinant 1, if $(c_1, d_1) = (c_2, d_2) \pmod{N}$, then the congruences (8.2.4–8.2.5) hold. Conversely, if (8.2.4–8.2.5) hold, then

$$\begin{aligned} c_2 &\equiv a_2 d_1 c_2 - b_2 c_1 c_2 \\ &\equiv a_2 d_2 c_1 - b_2 c_2 c_1 \qquad \text{since } d_1 c_2 \equiv d_2 c_1 \pmod{N} \\ &\equiv c_1 \qquad \text{since } a_2 d_2 - b_2 c_2 = 1, \end{aligned}$$

and likewise

$$d_2 \equiv a_2 d_1 d_2 - b_2 c_1 d_2 \equiv a_2 d_1 d_2 - b_2 d_1 c_2 \equiv d_1 \pmod{N}.$$

□

Thus we may view weight k Manin symbols for $\Gamma_1(N)$ as triples of integers (i, c, d), where $0 \leq i \leq k-2$ and $c, d \in \mathbb{Z}/N\mathbb{Z}$ with $\gcd(c, d, N) = 1$. Here (i, c, d) corresponds to the Manin symbol $[X^i Y^{k-2-i}, \left(\begin{smallmatrix} a & b \\ c' & d' \end{smallmatrix}\right)]$, where c' and d' lift c, d. The relations of Theorem 8.2.2 become

$$\begin{aligned} (i,\, c, d) + (-1)^i (k-2-i,\, d, -c) &= 0, \\ (i,\, c, d) \quad + \quad (-1)^{k-2} \sum_{j=0}^{k-2-i} (-1)^j \binom{k-2-i}{j} (j,\, d, -c-d) & \\ + \quad (-1)^{k-2-i} \sum_{j=0}^{i} (-1)^j \binom{i}{j} (k-2-i+j,\, -c-d, c) &= 0, \\ (i,\, c, d) - (-1)^{k-2} (i,\, -c, -d) &= 0. \end{aligned}$$

There is a similar description of cosets for $\Gamma_0(N)$:

Proposition 8.2.4. *The right cosets $\Gamma_0(N)\backslash \mathrm{SL}_2(\mathbb{Z})$ are in bijection with the elements of $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$. The coset containing a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ corresponds to the point $(c : d) \in \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$.*

For a proof, see [Cre97a, §2.2] (see also Exercise 3.3).

8.2.2 Modular Symbols With Character

Suppose now that $G = \Gamma_1(N) \subset \mathrm{SL}_2(\mathbb{Z})$. Define an action of diamond bracket operators $\langle d \rangle$, with $\gcd(d, N) = 1$ on Manin symbols as follows:

$$\langle n \rangle ([P, (c, d)]) = [P, (nc, nd)]\,.$$

Let

$$\varepsilon : (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{Q}(\zeta)^*$$

be a Dirichlet character, where ζ is an nth root of unity and n is the order of ε. Let $\mathbb{M}_k(\Gamma_1(N), \varepsilon)$ be the quotient of $\mathbb{M}_k(\Gamma_1(N), \mathbb{Z}[\zeta])$ by the relations (given in terms of Manin symbols)

$$\langle d \rangle x - \varepsilon(d)x = 0,$$

for all $x \in \mathbb{M}_k(\Gamma_1(N), \mathbb{Z}[\zeta])$, and by any torsion. Thus $\mathbb{M}_k(\Gamma_1(N), \varepsilon)$ is a torsion free $\mathbb{Z}[\varepsilon]$-module.

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1) Return all graded homework
2) Clarify current homework assignment:
     * All problems from dimension formulas chapter
       and one linear algebra problem:
          Exercise 7.1 (not easy).
       Should have more linalg exercises but too late.
     * Working together on homework is OK (I should
       have mentioned this earlier...)
3) Current plan for rest of quarter:
[] (may 15) Higher Weight Modular Symbols 1: basic definitions;
            how to compute:
             Sections 8.1 -- 8.2
[] (may 17) Higher Weight Modular Symbols 2: Hecke operators on them
             Sections 8.3
[] (may 19) Higher Weight Modular Symbols 3: using to compute modular forms
             Sections 8.4 -- 8.6
[] (may 22) Newforms 1: Atkin-Lehner-Li theory
             Sections 9.1 -- 9.3
[] (may 24) Newforms 2: Computing (and storing!) systems of eigenvalues
             Section 9.4
[] (may 26) Special Values of L-functions using modular symbols
             Sections 10.1-10.4
[] (may 29 -- memorial day holiday)
[] (may 31) Enumeration of all elliptic curves of given conductor
            (Cremona's program):
             Sections 10.6-10.7
[] (june 2) Sturm's bound: Congruences between modular forms and gen.
            Hecke algebras
             Chapter 11
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