Chapter 2

Modular Forms of Level 1

In this chapter we study in detail the structure of level 1 modular forms, i.e., modular forms on $\mathrm{SL}_2(\mathbb{Z}) = \Gamma_0(1) = \Gamma_1(1)$. We assume that you know some complex analysis (e.g., the residue theorem) and linear algebra, and have read Section 1.2.

2.1 Examples of Modular Forms of Level 1

In this section you will finally see some examples of modular forms of level 1! We will first introduce the Eisenstein series, one of each weight, then define Δ, which is a cusp form of weight 12. In Section 2.2 we will prove the structure theorem, which says that using addition and multiplication of these forms, we can generate all modular forms of level 1.

For an even integer $k \geq 4$, the *non-normalized weight k Eisenstein series* as a function on $\mathfrak{h}^*$ is

$$G_k(z) = \sum_{m,n\in\mathbb{Z}}^{*} \frac{1}{(mz+n)^k},$$

where for a given z, the sum is over all $m, n \in \mathbb{Z}$ such that $mz + n \neq 0$ (in particular, we omit nothing in the sum if $z \in \mathfrak{h}$).

Proposition 2.1.1. *The function $G_k(z)$ is a modular form of weight k, i.e., $G_k \in M_k(\mathrm{SL}_2(\mathbb{Z}))$.*

Proof. See [Ser73, § VII.2.3] for a proof that $G_k(z)$ defines a holomorphic function on $\mathfrak{h}^*$. To see that G_k is modular, observe that

$$G_k(z+1) = \sum^{*} \frac{1}{(m(z+1)+n)^k} = \sum^{*} \frac{1}{(mz+(n+m))^k} = \sum^{*} \frac{1}{(mz+n)^k},$$

where for the last equality we use that the map $(m, n+m) \mapsto (m, n)$ is invertible over $\mathbb{Z}$. Also,

$$\begin{aligned} G_k(-1/z) &= \sum^* \frac{1}{(-m/z+n)^k} \\ &= \sum^* \frac{z^k}{(-m+nz)^k} \\ &= z^k \sum^* \frac{1}{(mz+n)^k} = z^k G_k(z), \end{aligned}$$

where we use that $(n,-m) \mapsto (m,n)$ is invertible over $\mathbb{Z}$. □

Proposition 2.1.2. *$G_k(\infty) = 2\zeta(k)$, where ζ is the Riemann zeta function.*

Proof. In the limit as $z \to i\infty$ in the definition of $G_k(z)$, the terms involving z all go to 0 as $z \mapsto i\infty$. Thus

$$G_k(i\infty) = \sum_{n\in\mathbb{Z}}^* \frac{1}{n^k}.$$

This sum is twice $\zeta(k) = \sum_{n\geq 1} \frac{1}{n^k}$, as claimed. □

For example,

$$G_4(\infty) = 2\zeta(4) = \frac{1}{3^2\cdot 5}\pi^4$$

and

$$G_6(\infty) = 2\zeta(6) = \frac{2}{3^3\cdot 5\cdot 7}\pi^6.$$

2.1.1 The Cusp Form Δ

Suppose $E = \mathbb{C}/\Lambda$ is an elliptic curve over $\mathbb{C}$, viewed as a quotient of $\mathbb{C}$ by a lattice $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, with $\omega_1/\omega_2 \in \mathfrak{h}$. The *Weierstrass $\wp$-function* of the lattice Λ is

$$\wp = \wp_\Lambda(u) = \frac{1}{u^2} + \sum_{k=4,6,8,\ldots,\infty} (k-1)G_k(\omega_1/\omega_2)u^{k-2}.$$

It satisfies the differential equation

$$(\wp')^2 = 4\wp^3 - 60G_4(\omega_1/\omega_2)\wp - 140G_6(\omega_1/\omega_2).$$

If we set $x = \wp$ and $y = \wp'$ the above is an (affine) equation for an elliptic curve that is complex analytically isomorphic to $\mathbb{C}/\Lambda$. [[**Todo: See, e.g., Ahlfor's book.**]]

The discriminant of the cubic $4x^3 - 60G_4(\omega_1/\omega_2)x - 140G_6(\omega_1/\omega_2)$ is $16\Delta(\omega_1/\omega_2)$, where

$$\Delta = (60G_4)^3 - 27(140G_6)^2.$$

Since Δ is the difference of 2 modular forms of weight 12 it has weight 12. Morever,

$$\begin{aligned}\Delta(\infty) &= (60G_4(\infty))^3 - 27\,(140G_6(\infty))^2\\ &= \left(\frac{60}{3^2\cdot 5}\pi^4\right)^3 - 27\left(\frac{140\cdot 2}{3^3\cdot 5\cdot 7}\pi^6\right)^2\\ &= 0,\end{aligned}$$

so Δ is a cusp form of weight 12.

Lemma 2.1.3. *The only zero of the function Δ is at ∞.*

Proof. Let ω_1, ω_2 be as above. Since E is an elliptic curve, $\Delta(\omega_1/\omega_2) \neq 0$. □

2.1.2 Fourier Expansions of Eisenstein Series

Recall from (1.2.4) that elements f of $M_k(\mathrm{SL}_2(\mathbb{Z}))$ can be expressed as formal power series in terms of $q(z) = e^{2\pi i z}$, and that this expansion is called the Fourier expansion of f. The following proposition gives the Fourier expansion of the Eisenstein series $G_k(z)$.

Definition 2.1.4 (Sigma)**.** For any integer $t \geq 0$ and any positive integer n, let

$$\sigma_t(n) = \sum_{1\leq d|n} d^t$$

be the sum of the tth powers of the positive divisors of n. Also, let $\sigma(n) = \sigma_0(n)$, which is the number of divisors of n. For example, if p is prime then $\sigma_t(p) = 1 + p^t$.

Proposition 2.1.5. *For every even integer $k \geq 4$, we have*

$$G_k(z) = 2\zeta(k) + 2\cdot\frac{(2\pi i)^k}{(k-1)!}\cdot\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n.$$

Proof. See [Ser73, §VII.4], which uses a series of clever manipulations of series, starting with the identity

$$\pi\cot(\pi z) = \frac{1}{z} + \sum_{m=1}^{\infty}\left(\frac{1}{z+m} + \frac{1}{z-m}\right).$$

□

From a computational point of view, the q-expansion for G_k from Proposition 2.1.5 is unsatisfactory, because it involves transcendental numbers. For computational purposes, we introduce the *Bernoulli numbers* B_n for $n \geq 0$ *defined* by the following equality of formal power series:

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n\frac{x^n}{n!}. \tag{2.1.1}$$

Expanding the power series on the left we have

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \frac{x^6}{30240} - \frac{x^8}{1209600} + \cdots$$

As this expansion suggests, the Bernoulli numbers B_n with $n > 1$ odd are 0 (see Exercise 1.6). Expanding the series further, we obtain the following table:

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30},$$

$$B_{10} = \frac{5}{66}, \quad B_{12} = -\frac{691}{2730}, \quad B_{14} = \frac{7}{6}, \quad B_{16} = -\frac{3617}{510}, \quad B_{18} = \frac{43867}{798},$$

$$B_{20} = -\frac{174611}{330}, \quad B_{22} = \frac{854513}{138}, \quad B_{24} = -\frac{236364091}{2730}, \quad B_{26} = \frac{8553103}{6}.$$

See Section 2.7 for a discussion of fast (analytic) methods for computing Bernoulli numbers. Use the `bernoulli` command to compute Bernoulli numbers in SAGE.

```
sage: bernoulli(12)
-691/2730
sage: bernoulli(50)
495057205241079648212477525/66
sage: len(str(bernoulli(10000)))
27706
```

For us, the significance of the Bernoulli numbers is that they are rational numbers and they are connected to values of ζ at positive even integers.

Proposition 2.1.6. *If $k \geq 2$ is an even integer, then*

$$\zeta(k) = -\frac{(2\pi i)^k}{2 \cdot k!} \cdot B_k.$$

Proof. The proof in [Ser73, §VII.4] involves manipulating a power series expansion for $z\cot(z)$. □

Definition 2.1.7 (Normalized Eisenstein Series)**.** The *normalized Eisenstein series* of even weight $k \geq 4$ is

$$E_k = \frac{(k-1)!}{2 \cdot (2\pi i)^k} \cdot G_k$$

Combining Propositions 2.1.5 and 2.1.6 we see that

$$E_k = -\frac{B_k}{2k} + q + \sum_{n=2}^{\infty} \sigma_{k-1}(n) q^n. \tag{2.1.2}$$

It is thus now simple to explicitly write down Eisenstein series (see Exercise 2.1).