Homework 4: Primitive Roots and Quadratic Reciprocity

DUE WEDNESDAY, OCTOBER 17

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Math 124 HARVARD UNIVERSITY **Fall 2001**

1. (2 points) Calculate the following symbols by hand: $\left(\frac{3}{97}\right)$, $\left(\frac{5}{389}\right)$, $\left(\frac{2003}{11}\right)$, and $\left(\frac{5!}{7}\right)$.

2. (3 points) Prove that $\left(\frac{3}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1, 11 \pmod{12}, \\ -1 & \text{if } p \equiv 5, 7 \pmod{12}. \end{cases}$

3. (3 points) Prove that there is no primitive root modulo 2^n for any $n \geq 3$.

4. (6 points) Prove that if p is a prime, then there is a primitive root modulo p^2.

5. (5 points) Use the fact that $(\mathbb{Z}/p\mathbb{Z})^*$ is cyclic to give a direct proof that $\left(\frac{-3}{p}\right) = 1$ when $p \equiv 1 \pmod 3$. [Hint: There is an $c \in (\mathbb{Z}/p\mathbb{Z})^*$ of order 3. Show that $(2c+1)^2 = -3$.]

6. (6 points) If $p \equiv 1 \pmod 5$, show directly that $\left(\frac{5}{p}\right) = 1$ by the method of Exercise 5. [Hint: Let $c \in (\mathbb{Z}/p\mathbb{Z})^*$ be an element of order 5. Show that $(c + c^4)^2 + (c + c^4) - 1 = 0$, etc.]

7. (4 points) For which primes p is $\displaystyle\sum_{a=1}^{p-1} \left(\frac{a}{p}\right) = 0$?

8. (4 points) Artin conjectured that the number of primes $p \leq x$ such that 2 is a primitive root modulo p is asymptotic to $C\pi(x)$ where $\pi(x)$ is the number of primes $\leq x$ and C is a fixed constant called Artin's constant. Using a computer, make an educated guess as to what C should be, to a few decimal places of accuracy. Explain your reasoning. (Note: Don't try to prove that your guess is correct.)