is the space of holomorphic functions $f : \mathfrak{h} \to \mathbf{C}$ that satisfy the usual vanishing conditions at the cusps and such that for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(M, N)$,

$$f| \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \varepsilon(d) f.$$

We have

$$S_k(M, N) = \oplus_\varepsilon S_k(M, N, \varepsilon).$$

We now introduce operators between various $S_k(M, N)$. Note that, except when otherwise noted, the notation we use for these operators below is as in [Li75], which conflicts with notation in various other books. When in doubt, check the definitions.

Let

$$f| \begin{pmatrix} a & b \\ c & d \end{pmatrix} (\tau) = (ad - bc)^{k/2} (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right).$$

This is like before, but we omit the weight $k$ from the bar notation, since $k$ will be fixed for the whole discussion.

For any $d$ and $f \in S_k(M, N, \varepsilon)$, define

$$f|U_d^N = d^{k/2-1} f\Big| \left( \sum_{u \bmod d} \begin{pmatrix} 1 & uN \\ 0 & d \end{pmatrix} \right),$$

where the sum is over *any* set of representatives for the integers modulo $d$. Note that the $N$ in the notation is a superscript, not a power of $N$. Also, let

$$f|B_d = d^{-k/2} f| \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$f|C_d = d^{k/2} f| \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}.$$

In [Li75], $C_d$ is denoted $W_d$, which would be confusing, since in the literature $W_d$ is usually used to denote a completely different operator (the Atkin-Lehner operator, which is denoted $V_d^M$ in [Li75]).

Since $\left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma(M, N)$, any $f \in S_k(M, N, \varepsilon)$ has a Fourier expansion in terms of powers of $q_N = q^{1/N}$. We have

$$\left(\sum a_n q_N^n\right) |U_d^N = \sum_{n \geq 1} a_{nd} q_N^n,$$

$$\left(\sum a_n q_N^n\right) |B_d = \sum_{n \geq 1} a_n q_N^{nd},$$

and

$$\left(\sum a_n q_N^n\right) |C_d = \sum_{n \geq 1} a_n q_{Nd}^n.$$

The second two equalities are easy to see; for the first, write everything out and use that for $n \geq 1$, the sum $\sum_u e^{2\pi i u n/d}$ is 0 or $d$ if $d \nmid n$, $d \mid n$, respectively.

The maps $B_d$ and $C_d$ define injective maps between various spaces $S_k(M, N, \varepsilon)$. To understand $B_d$, use the matrix relation

$$\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & dy \\ z/d & w \end{pmatrix} \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix},$$

and a similar one for $C_d$. If $d \mid N$ then $B_d : S_k(M, N, \varepsilon) \to S_k(dM, N/d, \varepsilon)$ is an isomorphism, and if $d \mid M$, then $C_d : S_k(M, N) \to S_k(M/d, Nd, \varepsilon)$ is also an isomorphism. In particular, taking $d = N$, we obtain an isomorphism

$$B_N : S_k(M, N, \varepsilon) \to S_k(MN, 1, \varepsilon) = S_k(\Gamma_1(MN), \varepsilon). \tag{9.1.1}$$

Putting these maps together allows us to completely understand the cusp forms $S_k(\Gamma(N))$ in terms of spaces $S_k(\Gamma_1(N^2), \varepsilon)$, for all Dirichlet characters $\varepsilon$ that arise from characters modulo $N$. (Recall that $\Gamma(N)$ is the principal congruence subgroup $\Gamma(N) = \ker(\mathrm{SL}_2(\mathbf{Z}) \to \mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z}))$. This is because $S_k(\Gamma(N))$ is isomorphic to the direct sum of $S_k(N, N, \varepsilon)$, as $\varepsilon$ various over all Dirichlet characters modulo $N$.

For any prime $p$, the $p$th *Hecke operator* on $S_k(M, N, \varepsilon)$ is defined by

$$T_p = U_p^N + \varepsilon(p) p^{k-1} B_p.$$

Note that $T_p = U_p^N$ when $p \mid N$, since then $\varepsilon(p) = 0$. In terms of Fourier expansions, we have

$$\left( \sum a_n q_N^n \right) | T_p = \sum_{n \geq 1} \left( a_{np} + \varepsilon(p) p^{k-1} a_{n/p} \right) q_N^n,$$

where $a_{n/p} = 0$ if $p \nmid n$.

The operators we have just defined satisfy several commutativity relations. Suppose $p$ and $q$ are prime. Then $T_p B_q = B_q T_p$, $T_p C_q = C_q T_p$, and $T_p U_q^N = U_q^N T_p$ if $(p, qMN) = 1$. Moreover $U_d^N B_{d'} = B_{d'} U_d^N$ if $(d, d') = 1$.

*Remark* 9.1.1. Because of these relations, (9.1.1) describe $S_k(\Gamma(N))$ as a module over the ring generated by $T_p$ for $p \nmid N$.

**Definition 9.1.2** (Old Subspace)**.** The *old subspace* $S_k(M, N, \varepsilon)_{\mathrm{old}}$ is the subspace of $S_k(M, N, \varepsilon)$ generated by all $f|B_d$ and $g|C_e$ where $f \in S_k(M', N)$, $g \in S_k(M, N')$, and $M', N'$ are proper factors of $M$, $N$, respectively, and $d \mid M/M'$, $e \mid N/N'$.

Since $T_p$ commutes with $B_d$ and $C_e$, the Hecke operators $T_p$ all preserve $S_k(M, N, \varepsilon)_{\mathrm{old}}$, for $p \nmid MN$. Also, $B_N$ defines an isomorphism

$$S_k(M, N, \varepsilon)_{\mathrm{old}} \cong S_k(MN, 1, \varepsilon)_{\mathrm{old}}.$$

**Definition 9.1.3** (Petersson Inner Product)**.** If $f, g \in S_k(\Gamma(N))$, the *Petersson inner product* of $f$ and $g$ is

$$\langle f, g \rangle = \frac{1}{[\mathrm{SL}_2(\mathbf{Z}) : \Gamma(N)]} \int_D f(z) \overline{g(z)} y^{k-2} \, dx \, dy,$$

where $D$ is a fundamental domain for $\Gamma(N)$ and $z = x + iy$.

This Petersson pairing is normalized so that if we consider $f$ and $g$ as elements of $\Gamma(N')$ for some multiple $N'$ of $N$, then the resulting pairing is the same (since the volume of the fundamental domain shrinks by the index).

**Proposition 9.1.4** (Petersson)**.** *If $p \nmid N$ and $f \in S_k(\Gamma_1(N), \varepsilon)$, then $\langle f|T_p, g\rangle = \varepsilon(p)\langle f, g|T_p\rangle$.*

*Remark* 9.1.5. The proposition implies that the $T_p$, for $p \nmid N$, are diagonalizable. Be careful, because the $T_p$, with $p \mid N$, need not be diagonalizable.

**Definition 9.1.6** (New Subspace)**.** The *new subspace* $S_k(M, N, \varepsilon)_{\text{new}}$ is the orthogonal complement of $S_k(M, N, \varepsilon)_{old}$ in $S_k(M, N, \varepsilon)$ with respect to the Petersson inner product.

Both the old and new subspaces of $S_k(M, N, \varepsilon)$ are preserved by the Hecke operators $T_p$ with $(p, NM) = 1$.

*Remark* 9.1.7. Li [Li75] also gives a purely algebraic definition of the new subspace as the intersection of the kernels of various trace maps from $S_k(M, N, \varepsilon)$, which are obtained by averaging over coset representatives.

**Definition 9.1.8** (Newform)**.** A *newform* $f = \sum a_n q_N^n \in S_k(M, N, \varepsilon)$ is an element of $S_k(M, N, \varepsilon)_{\text{new}}$ that is an eigenform for all $T_p$, for $p \nmid NM$, and is normalized so that $a_1 = 1$.

Li introduces the crucial "Atkin-Lehner operator" $W_q^M$ (denoted $V_q^M$ in [Li75]), which plays a key roll in all the proofs, and is defined as follows. For a positive integer $M$ and prime $q$, let $\alpha = \text{ord}_q(M)$ and find integers $x, y, z$ such that $q^{2\alpha}x - yMz = q^\alpha$. Then $W_q^M$ is the operator defined by slashing with the matrix $\begin{pmatrix} q^\alpha x & y \\ Mz & q^\alpha \end{pmatrix}$. Li shows that if $f \in S_k(M, 1, \varepsilon)$, then $f|W_q^M|W_q^M = \varepsilon(q^\alpha)f$, so $W_q^M$ is an automorphism. Care must be taken, because the operator $W_q^M$ need not commute with $T_p = U_p^N$, when $p \mid M$.

After proving many technical but elementary lemmas about the operators $B_d$, $C_d$, $U_p^N$, $T_p$, and $W_q^M$, Li uses the lemmas to deduce the following theorems. The proofs are all elementary, but there is little I can say about them, except that you just have to read them.[1] 1

**Theorem 9.1.9.** *Suppose $f = \sum a_n q_N^n \in S_k(M, N, \varepsilon)$ and $a_n = 0$ for all $n$ with $(n, K) = 1$, where $K$ is a fixed positive integer. Then $f \in S_k(M, N, \varepsilon)_{\text{old}}$.*

From the theorem we see that if $f$ and $g$ are newforms in $S_k(M, N, \varepsilon)$, and if for all but finitely many primes $p$, the $T_p$ eigenvalues of $f$ and $g$ are the same, then $f - g$ is an old form, so $f - g = 0$, hence $f = g$. Thus the eigenspaces corresponding to the systems of Hecke eigenvalues associated to the $T_p$, with $p \nmid MN$, each have dimension 1. This is known as "multiplicity one".

**Theorem 9.1.10.** Let $f = \sum a_n q_N^n$ be a newform in $S_k(M, N, \varepsilon)$, $p$ a prime with $(p, MN) = 1$, and $q \mid MN$ a prime. Then

1. $f|T_p = a_p f$, $f|U_q^N = a_q f$, and for all $n \geq 1$,

$$a_p a_n = a_{np} + \varepsilon(p)p^{k-1}a_{n/p},$$
$$a_q a_n = a_{nq}.$$

[1] Remove from book.

If $L(f,s)=\sum_{n\geq 1} a_n n^{-s}$ is the Dirichlet series associated to $f$, then $L(f,s)$ has an Euler product

$$L(f,s)=\prod_{q\mid MN}(1-a_q q^{-s})^{-1}\prod_{p\nmid MN}(1-a_p p^{-s}+\varepsilon(p)p^{k-1}p^{-2s})^{-1}.$$

2. (a) If $\varepsilon$ is not a character mod $MN/q$, then $|a_q|=q^{(k-1)/2}$.
   (b) If $\varepsilon$ is a character mod $MN/q$, then $a_q=0$ if $q^2\mid MN$, and $a_q^2=\varepsilon(q)q^{k-2}$ if $q^2\nmid MN$.

## 9.2 The $U_p$ operator

Let $N$ be a positive integer and $M$ a divisor of $N$. For each divisor $d$ of $N/M$ we define a map

$$\alpha_d : S_k(\Gamma_1(M))\to S_k(\Gamma_1(N)) : \quad f(\tau)\mapsto f(d\tau).$$

We verify that $f(d\tau)\in S_k(\Gamma_1(N))$ as follows. Recall that for $\gamma=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$, we write

$$(f|[\gamma]_k)(\tau)=\det(\gamma)^{k-1}(cz+d)^{-k}f(\gamma(\tau)).$$

The transformation condition for $f$ to be in $S_k(\Gamma_1(N))$ is that $f|[\gamma]_k(\tau)=f(\tau)$. Let $f(\tau)\in S_k(\Gamma_1(M))$ and let $\iota_d=\left(\begin{smallmatrix} d & 0\\ 0 & 1\end{smallmatrix}\right)$. Then $f|[\iota_d]_k(\tau)=d^{k-1}f(d\tau)$ is a modular form on $\Gamma_1(N)$ since $\iota_d^{-1}\Gamma_1(M)\iota_d$ contains $\Gamma_1(N)$. Moreover, if $f$ is a cusp form then so is $f|[\iota_d]_k$.

**Proposition 9.2.1.** *If $f\in S_k(\Gamma_1(M))$ is nonzero, then*

$$\left\{\alpha_d(f)\;:\; d\mid \frac{N}{M}\right\}$$

*is linearly independent.*

*Proof.* If the $q$-expansion of $f$ is $\sum a_n q^n$, then the $q$-expansion of $\alpha_d(f)$ is $\sum a_n q^{dn}$. The matrix of coefficients of the $q$-expansions of $\alpha_d(f)$, for $d\mid (N/M)$, is upper triangular. Thus the $q$-expansions of the $\alpha_d(f)$ are linearly independent, hence the $\alpha_d(f)$ are linearly independent, since the map that sends a cusp form to its $q$-expansion is linear. □

When $p\mid N$, we denote by $U_p$ the Hecke operator $T_p$ acting on the image space $S_k(\Gamma_1(N))$. For clarity, in this section we will denote by $T_{p,M}$, the Hecke operator $T_p\in \mathrm{End}(S_k(\Gamma_1(M)))$. For $f=\sum a_n q^n\in S_k(\Gamma_1(N))$, we have

$$f|U_p=\sum a_{np}q^n.$$

Suppose $f=\sum a_n q^n\in S_k(\Gamma_1(M))$ is a normalized eigenform for all of the Hecke operators $T_n$ and $\langle n\rangle$, and $p$ is a prime that does not divide $M$. Then

$$f|T_{p,M}=a_p f \quad\text{and}\quad f|\langle p\rangle=\varepsilon(p)f.$$

Assume $N = p^r M$, where $r \geq 1$ is an integer. Let

$$f_i(\tau) = f(p^i\tau),$$

so $f_0, \ldots, f_r$ are the images of $f$ under the maps $\alpha_{p^0}, \ldots, \alpha_{p^r}$, respectively, and $f = f_0$. We have

$$\begin{aligned} f|T_{p,M} &= \sum_{n\geq 1} a_{np}q^n + \varepsilon(p)p^{k-1}\sum a_n q^{pn} \\ &= f_0|U_p + \varepsilon(p)p^{k-1}f_1, \end{aligned}$$

so

$$f_0|U_p = f|T_{p,M} - \varepsilon(p)p^{k-1}f_1 = a_p f_0 - \varepsilon(p)p^{k-1}f_1.$$

Also

$$f_1|U_p = \left(\sum a_n q^{pn}\right)|U_p = \sum a_n q^n = f_0.$$

More generally, for any $i \geq 1$, we have $f_i|U_p = f_{i-1}$.

The operator $U_p$ preserves the two dimensional vector space spanned by $f_0$ and $f_1$, and the matrix of $U_p$ with respect to the basis $f_0$, $f_1$ is

$$A = \begin{pmatrix} a_p & 1 \\ -\varepsilon(p)p^{k-1} & 0 \end{pmatrix},$$

which has characteristic polynomial

$$X^2 - a_p X + p^{k-1}\varepsilon(p). \qquad (9.2.1)$$

### 9.2.1 A Connection with Galois representations

This leads to a striking connection with Galois representations. Let $f$ be a newform and let $K = K_f$ be the field generated over $\mathbf{Q}$ by the Fourier coefficients of $f$. Let $\ell$ be a prime and $\lambda$ a prime lying over $\ell$. Then Deligne (and Serre, when $k = 1$) constructed a representation

$$\rho_\lambda : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}(2, K_\lambda).$$

If $p \nmid N\ell$, then $\rho_\lambda$ is unramified at $p$, so if $\mathrm{Frob}_p \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ if a Frobenius element, then $\rho_\lambda(\mathrm{Frob}_p)$ is well defined, up to conjugation. Moreover, one can show that

$$\begin{aligned} \det(\rho_\lambda(\mathrm{Frob}_p)) &= p^{k-1}\varepsilon(p), \quad \text{and} \\ \mathrm{tr}(\rho_\lambda(\mathrm{Frob}_p)) &= a_p. \end{aligned}$$

(We will discuss the proof of these relations further in the case $k = 2$.) Thus the characteristic polynomial of $\rho_\lambda(\mathrm{Frob}_p) \in \mathrm{GL}_2(E_\lambda)$ is

$$X^2 - a_p X + p^{k-1}\varepsilon(p),$$

which is the same as (9.2.1).

### *9.2.2 When is $U_p$ semisimple?*

**Question 9.2.2.** Is $U_p$ semisimple on the span of $f_0$ and $f_1$?

If the eigenvalues of $U_p$ are distinct, then the answer is yes. If the eigenvalues are the same, then $X^2 - a_p X + p^{k-1}\varepsilon(p)$ has discriminant 0, so $a_p^2 = 4p^{k-1}\varepsilon(p)$, hence

$$a_p = 2p^{\frac{k-1}{2}}\sqrt{\varepsilon(p)}.$$

**Open Problem 9.2.3.** Does there exist an eigenform $f = \sum a_n q^n \in S_k(\Gamma_1(N))$ such that $a_p = 2p^{\frac{k-1}{2}}\sqrt{\varepsilon(p)}$?

It is a curious fact that the Ramanujan conjectures, which were proved by Deligne in 1973, imply that $|a_p| \leq 2p^{(k-1)/2}$, so the above equality remains taunting. When $k = 2$, Coleman and Edixhoven proved that $|a_p| < 2p^{(k-1)/2}$.[2]

### *9.2.3 An Example of non-semisimple $U_p$*

Suppose $f = f_0$ is a normalized eigenform. Let $W$ be the space spanned by $f_0, f_1$ and let $V$ be the space spanned by $f_0, f_1, f_2, f_3$. Then $U_p$ acts on $V/W$ by $\overline{f}_2 \mapsto 0$ and $\overline{f}_3 \mapsto \overline{f}_2$. Thus the matrix of the action of $U_p$ on $V/W$ is $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, which is nonzero and nilpotent, hence not semisimple. Since $W$ is invariant under $U_p$ this shows that $U_p$ is not semisimple on $V$, i.e., $U_p$ is not diagonalizable.

## 9.3 The Cusp forms are free of rank one over $\mathbf{T}_{\mathbf{C}}$

### *9.3.1 Level 1*

Suppose $N = 1$, so $\Gamma_1(N) = \mathrm{SL}_2(\mathbf{Z})$. Using the Petersson inner product, we see that all the $T_n$ are diagonalizable, so $S_k = S_k(\Gamma_1(1))$ has a basis

$$f_1, \ldots, f_d$$

of normalized eigenforms where $d = \dim S_k$. This basis is canonical up to ordering. Let $\mathbf{T}_{\mathbf{C}} = \mathbf{T} \otimes \mathbf{C}$ be the ring generated over $\mathbf{C}$ by the Hecke operator $T_p$. Then, having fixed the basis above, there is a canonical map

$$\mathbf{T}_{\mathbf{C}} \hookrightarrow \mathbf{C}^d : \quad T \mapsto (\lambda_1, \ldots, \lambda_d),$$

where $f_i | T = \lambda_i f_i$. This map is injective and $\dim \mathbf{T}_{\mathbf{C}} = d$, so the map is an isomorphism of $\mathbf{C}$-vector spaces.

The form

$$v = f_1 + \cdots + f_n$$

generates $S_k$ as a $\mathbf{T}$-module. Note that $v$ is canonical since it does not depend on the ordering of the $f_i$. Since $v$ corresponds to the vector $(1, \ldots, 1)$ and $\mathbf{T} \cong \mathbf{C}^d$

[2] Look in Coleman-edixhoven and say more about this. Plus find the Weil reference. When $k = 2$, Weil [?] showed that $\rho_\lambda(\mathrm{Frob}_p)$ is semisimple, so if the eigenvalues of $U_p$ are equal then $\rho_\lambda(\mathrm{Frob}_p)$ is a scalar. But Edixhoven and Coleman [CE98] show that it is not a scalar by looking at the abelian variety attached to $f$.

acts on $S_k \cong \mathbf{C}^d$ componentwise, this is just the statement that $\mathbf{C}^d$ is generated by $(1, \ldots, 1)$ as a $\mathbf{C}^d$-module.

There is a perfect pairing $S_k \times \mathbf{T}_\mathbf{C} \to \mathbf{C}$ given by

$$\left\langle \sum f,\, T_n \right\rangle = a_1(f|T_n) = a_n(f),$$

where $a_n(f)$ denotes the $n$th Fourier coefficient of $f$. Thus we have simultaneously:

1. $S_k$ is free of rank 1 over $\mathbf{T}_\mathbf{C}$, and
2. $S_k \cong \mathrm{Hom}_\mathbf{C}(\mathbf{T}_\mathbf{C}, \mathbf{C})$ as $\mathbf{T}$-modules.

Combining these two facts yields an isomorphism

$$\mathbf{T}_\mathbf{C} \cong \mathrm{Hom}_\mathbf{C}(\mathbf{T}_\mathbf{C}, \mathbf{C}). \tag{9.3.1}$$

This isomorphism sends an element $T \in \mathbf{T}$ to the homomorphism

$$X \mapsto \langle v|T, X\rangle = a_1(v|T|X).$$

Since the identification $S_k = \mathrm{Hom}_\mathbf{C}(\mathbf{T}_\mathbf{C}, \mathbf{C})$ is canonical and since the vector $v$ is canonical, we see that the isomorphism (9.3.1) is canonical.

Recall that $M_k$ has as basis the set of products $E_4^a E_6^b$, where $4a + 6b = k$, and $S_k$ is the subspace of forms where the constant coefficient of their $q$-expansion is 0. Thus there is a basis of $S_k$ consisting of forms whose $q$-expansions have coefficients in $\mathbf{Q}$. Let $S_k(\mathbf{Z}) = S_k \cap \mathbf{Z}[[q]]$, be the submodule of $S_k$ generated by cusp forms with Fourier coefficients in $\mathbf{Z}$, and note that $S_k(\mathbf{Z}) \otimes \mathbf{Q} \cong S_k(\mathbf{Q})$. Also, the explicit formula $(\sum a_n q^n)|T_p = \sum a_{np} q^n + p^{k-1} \sum a_n q^{np}$ implies that the Hecke algebra $\mathbf{T}$ preserves $S_k(\mathbf{Z})$.

**Proposition 9.3.1.** *The Fourier coefficients of each $f_i$ are totally real algebraic integers.*

*Proof.* The coefficient $a_n(f_i)$ is the eigenvalue of $T_n$ acting on $f_i$. As observed above, the Hecke operator $T_n$ preserves $S_k(\mathbf{Z})$, so the matrix $[T_n]$ of $T_n$ with respect to a basis for $S_k(\mathbf{Z})$ has integer entries. The eigenvalues of $T_n$ are algebraic integers, since the characteristic polynomial of $[T_n]$ is monic and has integer coefficients.

The eigenvalues are real since the Hecke operators are self-adjoint with respect to the Petersson inner product. □

*Remark* 9.3.2. A *CM field* is a quadratic imaginary extension of a totally real field. For example, when $n > 2$, the field $\mathbf{Q}(\zeta_n)$ is a CM field, with totally real subfield $\mathbf{Q}(\zeta_n)^+ = \mathbf{Q}(\zeta_n + 1/\zeta_n)$. More generally, one shows that the eigenvalues of any newform $f \in S_k(\Gamma_1(N))$ generate a totally real or CM field.

**Proposition 9.3.3.** *We have $v \in S_k(\mathbf{Z})$.*

*Proof.* This is because $v = \sum \mathrm{Tr}(T_n) q^n$, and, as we observed above, there is a basis so that the matrices $T_n$ have integer coefficients. □

*Example* 9.3.4. When $k = 36$, we have

$$v = 3q + 139656q^2 - 104875308q^3 + 34841262144q^4 + 892652054010q^5 - 4786530564384q^6 + 878422149346056q^7 + \cdots .$$

The normalized newforms $f_1$, $f_2$, $f_3$ are

$$f_i = q + aq^2 + (-1/72a^2 + 2697a + 478011548)q^3 + (a^2 - 34359738368)q^4$$
$$(a^2 - 34359738368)q^4 + (-69/2a^2 + 14141780a + 1225308030462)q^5 + \cdots ,$$

for $a$ each of the three roots of $X^3 - 139656X^2 - 59208339456X - 1467625047588864$.

### *9.3.2 General level*

Now we consider the case for general level $N$. Recall that there are maps

$$S_k(\Gamma_1(M)) \to S_k(\Gamma_1(N)),$$

for all $M$ dividing $N$ and all divisor $d$ of $N/M$.

The *old subspace* of $S_k(\Gamma_1(N))$ is the space generated by all images of these maps with $M|N$ but $M \neq N$. The *new subspace* is the orthogonal complement of the old subspace with respect to the Petersson inner product.

There is an algebraic definition of the new subspace. One defines trace maps

$$S_k(\Gamma_1(N)) \to S_k(\Gamma_1(M))$$

for all $M < N$, $M \mid N$ which are adjoint to the above maps (with respect to the Petersson inner product). Then $f$ is in the new part of $S_k(\Gamma_1(N))$ if and only if $f$ is in the kernels of all of the trace maps.

It follows from Atkin-Lehner-Li theory that the $T_n$ acts semisimply on the new subspace $S_k(\Gamma_1(M))_{\mathrm{new}}$ for all $M \geq 1$, since the common eigenspaces for all $T_n$ each have dimension 1. Thus $S_k(\Gamma_1(M))_{\mathrm{new}}$ has a basis of normalized eigenforms. We have a natural map

$$\bigoplus_{M|N} S_k(\Gamma_1(M))_{\mathrm{new}} \hookrightarrow S_k(\Gamma_1(N)).$$

The image in $S_k(\Gamma_1(N))$ of an eigenform $f$ for some $S_k(\Gamma_1(M))_{\mathrm{new}}$ is called a *newform* of level $M_f = M$. Note that a newform of level less than $N$ is not necessarily an eigenform for all of the Hecke operators acting on $S_k(\Gamma_1(N))$; in particular, it can fail to be an eigenform for the $T_p$, for $p \mid N$.

Let

$$v = \sum_f f(q^{\frac{N}{M_f}}) \in S_k(\Gamma_1(N)),$$

where the sum is taken over all newforms $f$ of weight $k$ and some level $M \mid N$. This generalizes the $v$ constructed above when $N = 1$ and has many of the same good properties. For example, $S_k(\Gamma_1(N))$ is free of rank 1 over $\mathbf{T}$ with basis element $v$. Moreover, the coefficients of $v$ lie in $\mathbf{Z}$, but to show this we need to know that $S_k(\Gamma_1(N))$ has a basis whose $q$-expansions lie in $\mathbf{Q}[[q]]$. This is true, but we will not prove it here. One way to proceed is to use the Tate curve to construct a $q$-expansion map $\mathrm{H}^0(X_1(N), \Omega_{X_1(N)/\mathbf{Q}}) \to \mathbf{Q}[[q]]$, which is compatible with the usual Fourier expansion map.[3]

[3] Where will we?

*Example* 9.3.5. The space $S_2(\Gamma_1(22))$ has dimension 6. There is a single newform of level 11,

$$f = q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 + \cdots .$$

There are four newforms of level 22, the four $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$-conjugates of

$$g = q - \zeta q^2 + (-\zeta^3 + \zeta - 1)q^3 + \zeta^2 q^4 + (2\zeta^3 - 2)q^5 \\ + (\zeta^3 - 2\zeta^2 + 2\,\zeta - 1)q^6 - 2\zeta^2 q^7 + ...$$

where $\zeta$ is a primitive 10th root of unity.

*Warning* 9.3.6. Let $S = S_2(\Gamma_0(88))$, and let $v = \sum \mathrm{Tr}(T_n)q^n$. Then $S$ has dimension 9, but the Hecke span of $v$ only has dimension 7. Thus the more "canonical looking" element $\sum \mathrm{Tr}(T_n)q^n$ is not a generator for $S$. [4] 4

## 9.4 Decomposing the anemic Hecke algebra

We first observe that it make no difference whether or not we include the Diamond bracket operators in the Hecke algebra. Then we note that the $\mathbf{Q}$-algebra generated by the Hecke operators of index coprime to the level is isomorphic to a product of fields corresponding to the Galois conjugacy classes of newforms.

**Proposition 9.4.1.** *The operators $\langle d\rangle$ on $S_k(\Gamma_1(N))$ lie in $\mathbf{Z}[\ldots, T_n, \ldots]$.*

*Proof.* It is enough to show $\langle p\rangle \in \mathbf{Z}[\ldots, T_n, \ldots]$ for primes $p$, since each $\langle d\rangle$ can be written in terms of the $\langle p\rangle$. Since $p \nmid N$, we have that[5] 5

$$T_{p^2} = T_p^2 - \langle p\rangle p^{k-1},$$

so $\langle p\rangle p^{k-1} = T_p^2 - T_{p^2}$. By Dirichlet's theorem on primes in arithmetic progression [Lan94, VIII.4], there is another prime $q$ congruent to $p$ mod $N$. Since $p^{k-1}$ and $q^{k-1}$ are relatively prime, there exist integers $a$ and $b$ such that $ap^{k-1} + bq^{k-1} = 1$. Then

$$\langle p\rangle = \langle p\rangle(ap^{k-1} + bq^{k-1}) = a({T_p}^2 - T_{p^2}) + b({T_q}^2 - T_{q^2}) \in \mathbf{Z}[\ldots, T_n, \ldots].$$

□

Let $S$ be a space of cusp forms, such as $S_k(\Gamma_1(N))$ or $S_k(\Gamma_1(N), \varepsilon)$. Let

$$f_1, \ldots, f_d \in S$$

be representatives for the Galois conjugacy classes of newforms in $S$ of level $N_{f_i}$ dividing $N$. For each $i$, let $K_i = \mathbf{Q}(\ldots, a_n(f_i), \ldots)$ be the field generated by the Fourier coefficients of $f_i$.

---

[4] I think this because using my MAGMA program, I computed the image of v under $T_1$,...,$T_{25}$ and the span of the image has dimension 7. For example, there is an element of S whose q-expansion has valuation 7, but no element of the T-span of v has q-expansion with valuation 7 or 9.

[5] See where?

**Definition 9.4.2** (Anemic Hecke Algebra)**.** The *anemic Hecke algebra* is the subalgebra

$$\mathbf{T}_0 = \mathbf{Z}[\ldots, T_n, \ldots : (n, N) = 1] \subset \mathbf{T}$$

of $\mathbf{T}$ obtained by adjoining to $\mathbf{Z}$ only those Hecke operators $T_n$ with $n$ relatively prime to $N$.

**Proposition 9.4.3.** *We have* $\mathbf{T}_0 \otimes \mathbf{Q} \cong \prod_{i=1}^{d} K_i$.

The map sends $T_n$ to $(a_n(f_1), \ldots, a_n(f_d))$. The proposition can be proved using the discussion above and Atkin-Lehner-Li theory, but we will not give a proof here.[6] 6

*Example* 9.4.4. When $S = S_2(\Gamma_1(22))$, then $\mathbf{T}_0 \otimes \mathbf{Q} \cong \mathbf{Q} \times \mathbf{Q}(\zeta_{10})$ (see Example 9.3.5). When $S = S_2(\Gamma_0(37))$, then $\mathbf{T}_0 \otimes \mathbf{Q} \cong \mathbf{Q} \times \mathbf{Q}$.

[6] Add for book.