Harvard Math 129: Algebraic Number Theory

Midterm

William Stein

Due: 10:10am in class on Thursday, March 17, 2005

This midterm is worth 25% of your grade. There are four problems and this exam is out of 100 points. The point value of each problem is as indicated.

Rules: *You may use your course notes and textbook, but no other books, the internet, or people. You may use math software, including* Magma *and* PARI*, though I don't think you will need to for any of these problems, since they are all conceptual.*

1. (20 points) Let R be a noetherian integral domain, and let $K = \mathrm{Frac}(R)$ be the field of fractions of R. Let $\overline{K}$ be an algebraic closure of K. Let $\overline{R}$ be the set of $\alpha \in \overline{K}$ such that there is a nonzero monic polynomial $f(x) \in R[x]$ with $f(\alpha) = 0$. Is $\overline{R}$ necessarily a ring? Prove or give a counterexample.

2. (10 points) Let $\mathcal{O}_K$ be the ring of integers of a number field K, and suppose K has exactly $2s$ complex embeddings. Prove that the sign of the discriminant of $\mathcal{O}_K$ is $(-1)^s$.

3. Suppose K is a number field. For any finite extension L of K, define set-theoretic maps

$$\begin{aligned} \Psi_L : C_K \to C_L, \qquad & [I] \mapsto [I\mathcal{O}_L] \\ \Phi_L : C_L \to C_K, \qquad & [I] \mapsto [I \cap \mathcal{O}_K], \end{aligned}$$

where $[I]$ denotes the class of the nonzero integral ideal I.

(a) (10 points) Is Ψ_L a group homomorphism? Prove or give a counterexample.

(b) (10 points) Is Φ_L a group homomorphism? Prove or give a counterexample.

(c) (20 points) Prove that there is a number field L such that Ψ_L is the 0 map, i.e., Ψ_L sends every element of C_K to the identity of C_L. [Hint: Use finiteness of C_K in two ways.]

(Note: I wonder, is there always an L such that Φ_L is the 0 map? This question just occured to me while writing this exam. If you find an answer and tell me the answer, I'll be very thankful, though this is not part of the official exam. This question is related to "visibility of Mordell-Weil groups", which I just wrote a paper about.)

4. A number field is *totally real* if every embedding is real, i.e., $s = 0$, and a number field is *totally complex* if every embedding is complex, i.e., $r = 0$.

   (a) (15 points) Find with proof the possible degrees of totally real fields.

   (b) (15 points) Find with proof the possible degrees of totally complex fields.