Bernoulli numbers and the unity of mathematics.

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(A handout for the Hilldale Lecture.)

Here are the first few Bernoulli numbers referred to in the title, dripping down the left hand side of the page.

$$
\begin{aligned}
B_0 &= 1\\
B_1 &= -1/2\\
B_2 &= 1/6\\
B_4 &= -1/30\\
B_6 &= 1/42\\
B_8 &= -1/30\\
B_{10} &= 5/66\\
B_{12} &= -691/2730\\
B_{14} &= 7/6\\
B_{16} &= -3617/510\\
B_{18} &= 43867/798\\
B_{20} &= -174611/330\\
B_{22} &= 854513/138\\
B_{24} &= -23634091/2730\\
B_{26} &= 8553103/6\\
B_{28} &= -23749461029/870\\
B_{30} &= 8615841276005/14322\\
B_{32} &= -7709321041217/510\\
&\dots
\end{aligned}
$$

These Bernoulli numbers are rational numbers. You'll notice that except for B_1 the odd number indices are missing as entries of the above list. This is because $B_k = 0$ for $k > 1$ an odd number. Also the even-indexed Bernoulli numbers alternate in sign.

People who work with these numbers sometimes make personal attachments to them; for example, my favorites in this list are B_{12} and B_{32} (in that order). We'll see why, in the lecture.

You might wonder how a mere sequence of rational numbers can possibly be a "unifying force" in mathematics as the title of my lecture is meant to suggest. *Theories*, of course, can unify: *category theory*, for example, or *set theory*; physicists have their quest for a “unified theory of everything.” But how can a bunch of numbers have the effect of unifying otherwise seemingly disparate branches of our subject?

As we’ll see, for starters, Bernoulli numbers sit in the center of a number of mathematical fields, and whenever, for a given index k the Bernoulli number B_k exhibits some particular behavior, these different mathematical fields seem to feel the consequences, each in their own way.

The “Bernoulli Number” Website http://www.mscs.dal.ca/ dilcher/bernoulli.html offers a bibliography of a few thousand articles giving us a sense*that* these numbers pervade mathematics, but to get a more vivid sense of *how* they do so, we will survey, in the lecture, the pertinence of Bernoulli numbers in just a few subjects.

There may have been early appearances of the sequence of numbers referred to as Bernoulli numbers, but it is traditional to think of them as originating in Jacob Bernoulli’s posthumous manuscript Conjectandi (published 1713).

The text *Ars Conjectandi* itself might stand for the unity inherent in mathematics. It ostensibly focusses on *combinatorics* which, as Bernoulli says, corrects our most frequent error (counting things incorrectly) and is an art “most useful, because it remedies this defect of our minds and teaches how to enumerate all possible ways in which several things can be combined, transposed, or joined with another.”* Bernoulli continues by claiming that this art is so important that

“neither the wisdom of the philosopher nor the exactitude of the historian, nor the dexterity of the physician, nor the prudence of the statesman can stand without it.”

He goes on to say that the work of these people depend upon ” *conjecturing* and every conjecture involves weighing complexions or combinations of causes.” For Bernoulli, *conjecturing* means quantitatively assessing the likelihood of an outcome, given one’s current partial knowledge; in other words, “figuring the odds.” Indeed *Ars Conjectandi* is viewed as one of the founding texts in probability, but it roams wide. For example, Bernoulli’s notion of probability, including the famous law of large numbers whose origin is in this treatise, is not entirely without theological overtones. Bernoulli suggests by some of his terminology that, in his view, the law exhibits an overarching sense of *pre-destination*, for events are constrained to occur in specific ironclad frequencies, even though, from our finite viewpoint, it might appear as if things were random. Here is how *Ars Conjectandi* ends:

“Whence at last this remarkable result is seen to follow, that if the observations of all events were continued for the whole of eternity (with the probability finally transformed

* I am thankful to Edith Sylla for providing me with a manuscript of her new English translation (in progress) of Bernoulli’s treatise; all quotations from that treatise, given below, are from her translation.

into perfect certainty) then everything in the world would be observed to happen in fixed ratios and with a constant law of alternation. Thus in even the most accidental and fortuitous we would be bound to acknowledge a certain quasi necessity and, so to speak, fatality. I do not know whether or not Plato already wished to assert this result in his dogma of the universal return of things to their former positions [apocatastasis], in which he predicted that after the unrolling of innumerable centuries everything would return to its original state."

Bernoulli initiates his discussion, though, by concentrating on the combinatorics of what we call *binomial coefficients*–i.e., "Pascal's triangle,"–and what he calls his table of "figurate numbers."** He writes:

"This Table has clearly admirable and extraordinary properties, for beyond what I have already shown of the mystery of combinations hiding within it, it is known to those skilled in the more hidden parts of geometry that the most important secrets of all the rest of mathematics lie concealed within it."

This, of course, is a serious claim.

The numbers that will eventually be attached to his name enter Bernoulli's treatise only briefly, and in the discussion of closed forms for the sums of k-th powers of consecutive integers.

The Bernoulli numbers in question are the coefficients of the linear terms of these polynomial expressions. His predecessors had already made some computations of the polynomials. In particular, Johann Faulhaber (1580-1635) of Ulm computed the formulas up to $k = 17$ in his *Mysterium Arithmeticum* published in 1615. But Bernoulli chides them (Wallis included) for first laboriously working out closed expressions for the sums of consecutive k-th powers and then trying to understand "figurate numbers" in terms of these formulas, rather than what Bernoulli himself does which is to reverse the procedure; namely, he bases his analysis on the formula

$$\sum_{k=1}^{n-1} \frac{k\cdot(k-1)\cdot\ldots(k-c+1)}{1\cdot 2\cdot\ldots(c-1)} \;=\; \frac{(n\cdot(n-1)\cdot\ldots(n-c)}{1\cdot 2\cdot\ldots(c)}$$

and he derives the formulas for power sums from this, and then goes on to explain why this is philosophically, as well as practically, the better method.

He proclaims that one can continue his table without, as he puts it, "digressions," by deriving the basic formula that he writes as

$$(*) \qquad \sum n^c \;=\; \frac{1}{c+1}n^{c+1} + \frac{1}{2}n^c + \frac{c}{2}An^{c-1} + \frac{c(c-1)(c-2)}{4!}Bn^{c-3} +$$

** The terminology *figurate numbers* takes off from the fact that the numbers $\frac{n\cdot(n-1)}{2}$ are *triangular numbers*; i.e., they count the number of dots in an orderly array forming a right-angle triangle. Similarly the higher binomial coefficients fill out elementary polytopes in higher dimensions.

$$+\frac{c(c-1)(c-2)(c-3)(c-4)}{6!}Cn^{c-5}+\frac{c(c-1)(c-2)(c-3)(c-4)(c-5)(c-6)}{8!}Dn^{c-7}+\ldots$$

where $A=\frac{1}{6}, B=-\frac{1}{30}, C=\frac{1}{42}, D=-\frac{1}{30},\ldots$

These $A,B,C,D,\ldots$ of course, are the numbers $B_2,B_4,B_6,B_8,\ldots$ that will bear his name. Bernoulli explains how to rapidly compute them (specifically, by induction: for example if you know A,B,C you can get D by setting $n=1$ and $c=8$ in the above formula, etc.). Bernoulli has sketched, in effect, a recurrent procedure for calculating the $B_k's$ but there is no difficulty producing some straight "explicit formulas" such as:

$$B_k \;=\; \frac{(-1)^k k}{2^k-1}\sum_{i=1}^{k} 2^{-i}\sum_{j=0}^{i-1}(-1)^j\binom{i-1}{j}(j+1)^{k-1}.$$

This formula was published some 170 years after *Ars Conjectandi* by J. Worpitsky (for the history and the derivation of this and other explicit formulas, see articles by H.W. Gould, *Explicit formulas for Bernoulli numbers* American Mathematical Monthly, **79** (1972) 44-51, and G. Rządkowsi, *A short proof of the Explicit Formula for Bernoulli numbers*, American Mathematical Monthly, **111** (2004) 432-434). More telling for our story is the standard definition given nowadays. Namely, the Bernoulli numberB_k is the coefficient of $\frac{x^k}{k!}$ in the power series expansion

$$\frac{x}{e^x-1} \;=\; 1-\frac{x}{2}+\sum_{k=2}^{\infty} B_k\frac{x^k}{k!}.$$

Bernoulli was proud of his recursive procedure and was not above taunting his predecessors:

"I have found in less than a quarter of an hour that the tenth powers (or the *quadrate-sursolids*) of the first thousand numbers beginning from 1 added together equal

$$91,409,924,241,424,243,424,241,924,242,500,$$

from which it is apparent how useless should be judged the works of Ismael Bullialdus, recorded in the thick volume of his *Arithmeticae Infinitorum*, where all he accomplishes is to show that with immense labor he can sum the first six powers–part of what we have done in a single page."

With that salvo, Bernoulli makes no further mention, in his treatise, of the numbers we will be concentrating on, and turns his attention to other things. But now it is time for us to examine exactly how the Bernoulli numbers act as a unifying force, holding together seemingly disparate fields of mathematics ...