How Explicit is the Explicit Formula?

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–William Stein. Joint work with Barry Mazur.
(and some with Hao _)

① Birch & Swinnerton-Dyer's Hunch.

$E/\mathbb{Q}$ elliptic curve

$E(\mathbb{Q}) \rightsquigarrow E(\mathbb{F}_2), E(\mathbb{F}_3), \ldots E(\mathbb{F}_p)$

$\mathrm{sign}(a_p) = \begin{cases} 1 & \text{if } E(\mathbb{F}_p) \text{ small} \\ +1 & \text{if } E(\mathbb{F}_p) \text{ big} \end{cases}$

big $\xLeftrightarrow[\text{BIAS}]{\text{STATISTICAL}}$ tend to be big

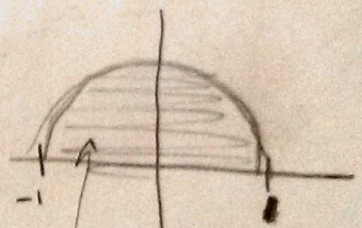
$a_p = p+1 - \#E(\mathbb{F}_p)$ "big" means $a_p < 0$.

$\mathrm{rank}(E(\mathbb{Q})) \overset{\text{conj}}{=} \mathrm{ord}_{s=1} L(E,s)$, $L^*(E,s) = \prod \frac{p}{1 - a_p p^{-s} + p^{1-2s}}$, $L^*(E,1) \text{"="} \prod \frac{1}{\#E(\mathbb{F}_p)}$

Theorem ("Sato-Tate", Taylor et al.) (E not CM)

$\left\{ \frac{a_p}{2\sqrt{p}} : p \text{ prime} \right\}$ is distributed like this

-1 ≤



perfect semicircle!
(show picture for $p < 10^9$).
Animation!
for 5077a.

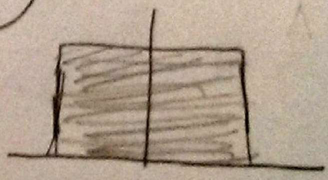
So: in the limit $\mathrm{sign}(a_p) = \pm 1$ with equal prob!

② Chebyschev Bias:

This talk: Explain what is really going on here.

$2 \neq p$ prime $\qquad p \equiv \begin{cases} 1 \\ 3 \end{cases} \pmod 4$

Theorem (Dirichlet): Distribution of $p \bmod 4$ is this



(show computer pic for $p < 10^9$)
Animation.

Chebyshev more refined question: Dirichlet coef. of $L(\chi_4, s)$

What about $\delta(x) = \sum_{2 < p \leq X} \left(\frac{-1}{p}\right)$ where $b_p = \left(\frac{-1}{p}\right) = \begin{cases} -1 & \text{if } p \equiv 3 \\ 1 & \text{if } p \equiv 1 \end{cases}$?

Prime race: show an animation.

Theorem (Rubinstein–Sarnak):

③ The Explicit Formula (Riemann)

(Sum of local data) = (Global data) + (Oscillatory term) + (convergent term)

$S(X) = F(X) \sum_{p \le X} G(p)$ (real zeros of L-functions (so BSD, etc!)) (Infinite sum over imaginary zeros (so GRH)) (?)

Theorem (Riemann) $\sum_{p<X} 1$.

$\Lambda(X) = \text{Von Mangoldt} = \begin{cases} 0 & \text{otherwise,} \\ \log(p) & \text{if } X = p^n. \end{cases}$

$\Psi_0(X) = \frac{1}{2}\Lambda(X) + \sum_{n<X} \Lambda(n)$

Then

$$\frac{1}{X}\Psi_0(X) = 1 - \sum_{|\gamma| \le T} \frac{X^{i\gamma}}{(\frac{1}{2}+i\gamma)\sqrt{X}} + C(X,T)$$

zeros of $\zeta(s)$

(a statement for each T)

$\lim_{X\to\infty} \lim_{T\to\infty} C(X,T) = 0$

where: $C(X,T) = \frac{-\log(2\pi) - \log(1 - 1/X^2)/2}{X} + \varepsilon(X,T)$ (Plot this all)

$|\varepsilon(X,T)| \ll \frac{\log(X)}{X} \cdot \min\left(1, \frac{X}{T\langle X\rangle}\right) + \frac{\log^2(XT)}{T}$.

④ Raw, medium, and well-done sums — elliptic curves $E/\mathbb{Q}$ (or # field) or newform

Raw data:

$$\Delta_E(X) = \frac{\log X}{\sqrt{X}} \sum_{p \le X} \operatorname{sign}(a_E(p))$$

Plot each for $p \le 10^9$ on a log scale for several E.

$\boxed{\text{Random walks}}$ Histogram of "returns" is Sato-Tate

Medium-rare data:

$$\mathfrak{D}_E(X) = \frac{\log(X)}{\sqrt{X}} \sum_{p \le X} \frac{a_E(p)}{\sqrt{p}}$$

Plot frequency histogram of values of each.

Well-done data:

$$D_E(X) = \frac{1}{\log(X)} \sum_{p \le X} \frac{a_E(p) \cdot \log(p)}{p}$$

- raw: infinite variance
- medium: finite variance
- well-done: nice.

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⑤ Sarnak's Letter – conjectures

Make histogram idea precise:

If $X \mapsto \delta(X)$ continuous function on $\mathbb{R}$, say $\delta(x)$ possesses limiting distribution μ_δ wrt multiplicative measure $\frac{dx}{x}$, if for bounded cont f:

$$\lim_{X\to\infty} \frac{1}{\log X}\int_0^X f(\delta(x))\, dx/x = \int_{\mathbb{R}} f(x)\, d\mu_\delta(x).$$

Mean:

$$\mathcal{E}(\delta(x)) = \int_{\mathbb{R}} d\mu_\delta(x) = \lim_{X\to\infty} \frac{1}{\log X}\int_0^X \delta(x)\frac{dx}{x}.$$

so $f=1$

Show plot of for each on log scale.

Conj (Sarnak):

- $\mathcal{E}(D_E) = -r = -(\text{analytic rank of } E)$.
- $\mathcal{E}(\delta_E) = 1-2r$
- $\mathcal{E}(\Delta_E) = \frac{2}{\pi} - \frac{16}{3\pi}r + \frac{4}{\pi}\sum_{k=1}^{\infty}(-1)^{k+1}\left[\frac{1}{2k+1} + \frac{1}{2k+3}\right] r_E(2k+1)$.

He deduces these using GRH + more + explicit formula. $r(n) = \mathrm{ord}\, L(\mathrm{Sym}^n E)$ s=center.

⑥ Data. A Minimalist Conjecture.

Conj: 50% of elliptic curves have rank 0
50% ——— 1.

Data (which is against this) RM heuristics – for this

Open Question: What about $r_E(n)$? Minimal 100% of time subject to f.e.?

Very little data — Watkins – ___ : ___ curves.

Idea: Assume minimalist and "compute" $\mathcal{E}(\Delta_E)$...

Examples: (various ranks)

⑦ Oscillatory Term, The Bayesian

Explicit formula for $D_E(X)$ (well-done):

$$D_E(X) = -r + \underbrace{\frac{1}{\log X} \cdot \underbrace{\sum_{0<|\gamma|\leq T} \frac{X^{i\gamma}}{i\gamma}}_{S(X,T)}}_{\text{goes to } 0} + \underbrace{C_E(X,T)}_{\text{goes to } 0}$$

What is $S(X,T)$? Crazy.

$\parallel$ calculation

$$\frac{2}{\log(X)} \sum_{0<\gamma\leq T} \frac{1}{\gamma} \sin(\gamma \log X)$$

- sort of alternating harmonic series.
- locally crazy as $T \to \infty$, for X in interval
- plot it.

The Bayesian: . Frequency histogram of $S(X,T)$ (fixed T, $X \to \infty$)

my idea

Conj: Values are normally distributed: show examples

So we "compute" the dist, hence understand that term.

⑧ The convergent Term.

Show some plots.

I don't understand $C_E(X,T)$ yet. but it might be the easiest (or hardest) to