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Inventiones mathematicae

Nonvanishing theorems for L-functions of modular forms and their derivatives*

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0. Introduction

Let M be a positive integer. If f is a normalized newform for $\Gamma_0(M)$, then the L-function with gamma factors attached has the form

$$\Lambda(s,f) = (2\pi)^{-s}\Gamma(s)M^{s/2}L(s,f)\,.$$

It satisfies a functional equation $\Lambda(s,f) = \varepsilon\Lambda(k-s,f)$, where $\varepsilon = \pm 1$ is naturally called the *sign in the functional equation.* Clearly the order of vanishing of $L(s,f)$ at $s = k/2$ is even if the sign in the functional equation is positive, and is odd if the sign in the functional equation is negative.

If D is a fundamental discriminant, let χ_D be the Dirichlet character with conductor $|D|$ defined in terms of the Kronecker symbol by $\chi_D(n) = \left(\frac{D}{n}\right)$. This is the quadratic character associated with the quadratic field $\mathbf{Q}(\sqrt{D})$. It is known that $\chi_D(-1) = \operatorname{sgn}(D)$.

Let $L(s,f,\chi_D) = \sum \chi_D(n)a(n)n^{-s}$, and let

$$\Lambda(s,f,\chi_D) = (D^2M)^{s/2}(2\pi)^{-s}\Gamma(s)L(s,f,\chi_D)\,.$$

This has analytic continuation as a function of s, and, provided $\gcd(M,D) = 1$, satisfies the functional equation

$$\Lambda(s,f,\chi_D) = \varepsilon\chi_D(-M)\Lambda(k-s,f,\chi_D)\,.$$

Now suppose that *every prime dividing M splits in $\mathbf{Q}(\sqrt{D})$*. In terms of the quadratic symbol, this implies that $\chi_D(M) = 1$. Therefore the sign in the functional equation for $L(s,f,\chi_D)$ is the same as the sign in the functional equation for $L(s,f)$ if $D > 0$, and opposite if $D < 0$.

Theorem. *Let f be cuspidal newform of even weight k with trivial character for the group $\Gamma_0(M)$, and let S be a finite set of primes including all those dividing M. Let ε denote the sign in the functional equation of f.*

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(i) *There exists a quadratic field* $\mathbf{Q}(\sqrt{D})$ *with* D *a fundamental discriminant, and* $\varepsilon D < 0$, *such that every prime in* S *splits in* $\mathbf{Q}(\sqrt{D})$, *and* $L(s, f, \chi_D)$ *has a simple zero at* $s = k/2$.

(ii) *There exists a quadratic field* $\mathbf{Q}(\sqrt{D})$ *with* D *a fundamental discriminant, and* $\varepsilon D > 0$, *such that every prime in* S *splits in* $\mathbf{Q}(\sqrt{D})$, *and* $L(k/2, f, \chi_D) \neq 0$.

It is explained in [4] that (i), combined with work of Kolyvagin [13] and of Gross and Zagier [7], implies that if E is a modular elliptic curve over $\mathbf{Q}$ such that $L(1, E) \neq 0$, then the group of rational points of E, and also the Tate-Shafarevich group of E over $\mathbf{Q}$, are finite. A result similar to (i) has been obtained by a different method by M. R. Murty and V. K. Murty [15].

The second assertion was proved by a different technique by Waldspurger [19] and [20]. See also Kohnen [12] and Jacquet [10].

In the proof of the Theorem we may assume that $\varepsilon = +1$. For if the sign in the functional equation of f is negative, we may twist f by an odd quadratic character whose ramification is outside of S to obtain a form the sign in whose functional equation is positive, and then apply the Theorem to the twisted form (cf. Atkin and Li [1]). *We will therefore assume that the sign in the functional equation of* f *is positive.*

The method of the proof is that introduced in [3]. We consider a Dirichlet series in a new parameter u, whose D-th coefficient is $L(s, f, \chi_D)$, by applying a type of integral transform introduced (in a different context) by Novodvorsky [15] to an Eisenstein series of half-integral weight associated with f on the metaplectic group (the double cover of $GSp(4, \mathbf{R})$). Actually there are two such Dirichlet series, corresponding to positive and negative discriminants. We obtain our results by studying the poles in u of these Dirichlet series.

An important technical difference between this work and [3] comes from the decision to use *Jacobi modular forms*. This was an excellent suggestion made to us by D. Zagier. Jacobi modular forms are automorphic forms on the Jacobi group, which in our situation is the semidirect product of $GSp(4, \mathbf{R})$ and a Heisenberg group. The Eisenstein series of half-integral weight previously alluded to occur as coefficients in Fourier-Jacobi expansions of these Jacobi Eisenstein series. In practical terms, this means that the "theta multiplier"—the automorphy factor for the Eisenstein series—is built into the framework of our work, and does not require special attention. Secondly, it means that the method by which we compute the Whittaker-Fourier coefficients of the Eisenstein series is rather different from that used in [3]. Thirdly, the problem of sifting out just those quadratic characters χ_D such that the primes in S split in $\mathbf{Q}(\sqrt{D})$ is solved very felicitously if one uses Jacobi forms. Naturally, it would have been possible to carry out our program without using Jacobi modular forms, as we did in [3]. However, it seems clear that the work would not have been as elegant.

As in [3], a technical problem is presented by the fact that two poles, corresponding to two different cells in the Bruhat decomposition on $GSp(4)$, coalesce when $s = k/2$. The contribution to one pole is very explicitly computed. The other pole comes with a factor which is a Dirichlet series with an Euler product. If $p \notin S$ the p-factor of this Euler product is explicitly computable. However, if $p \in S$, the p-factor of this Euler product is more complex. Fortunately, it is not necessary for us to compute these "ramified" Euler factors explicitly, for it turns out that there is enough extra structure in the situation that we are able to understand the interaction of the two poles. In other words, *because of this extra structure we are able to completely avoid dealing with problems of ramification.*

Another technical problem which does *not* occur in [3] (where we chose to work over the field $\mathbf{Q}(i)$ in order to achieve technical simplifications) arises from the necessity of separating the contributions of the positive and negative discriminants. Although we did not have to deal with this problem in [3], it did arise in the work of Goldfeld and Hoffstein [6], where quadratic twists of Dirichlet L-functions were considered. A similar strategy was used in that work to that of the current paper. The role played by the Eisenstein series on the double cover of $GSp(4, \mathbf{R})$ in the current paper was played, in the work of Goldfeld and Hoffstein, by the Eisenstein series on the double cover of $SL(2, \mathbf{R})$. In both cases, the key to separating the contributions to the two sign classes of discriminants involves varying the K-type of the Eisenstein series, i.e. the representation of the maximal compact subgroup K which is built into the definition of the Eisenstein series. In [6], the maximal compact subgroup of $SL(2, \mathbf{R})$ is the abelian group $SO(2)$. The K-type of the Eisenstein series is simply the weight. It was shown that the Mellin transforms of the two Whittaker functions associated with positive and negative discriminants had different asymptotics as the weight was varied. In the current work, the maximal compact subgroup K of $GSp(4, \mathbf{R})$ is $U(2)$, which is a somewhat richer group. The possibility of varying the K-type of the Eisenstein series is manifested by the Peter-Weyl theorem, which says that any continuous function on K may be uniformly approximated by a matrix coefficient of a representation. This allows us to build into the Whittaker functions which occur a rather arbitrary function on K, giving us considerable flexibility.

It is clear that the techniques of the present paper can be used to obtain mean value estimates analogous to those of [6], for quadratic twists of automorphic L-functions and their derivatives. (Cf. [4] for a precise statement.)

Although this work may seem technically imposing, we would like to stress that there is an essential underlying simplicity. Most of the work involved consists in laying the foundations for the theory of Eisenstein series and Whittaker functions on the metaplectic group. Once the foundations are laid, the structure of the proof is extremely simple.

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1. Jacobi modular forms

Basic references for Jacobi modular forms are the work of Eichler and Zagier [5], and of Ziegler [22].

As in the Theorem, let f be a newform for $\Gamma_0(M)$, the sign in whose functional equation is positive. We will actually wish to work with a form of higher level. We therefore regard f as an oldform for the group $\Gamma_0(N)$ where N is a suitable multiple of M. We require that N be divisible by 8, and by every prime in the set S of the Theorem. We will also require an integer m, such that $N|m$ and $4m|N^2$. In the final section we will specify m more precisely. With these preliminaries, we will now set out to define a *Jacobi modular Eisenstein series* which is built around the form f. In the next section, we will consider Fourier-Jacobi expansions of Jacobi forms. For the Eisenstein series in question, the coefficients in these expansions will involve certain *Eisenstein series of half-integral weight*, which are the ultimate objects of concern.

If z is a complex number, then $e(z)$ will denote $e^{2\pi iz}$. We will denote by $e^m(z)$ the function $e^{2\pi imz}$.

If A is a matrix, then ${}^{\mathrm{T}}A$ will denote its transpose. If R is any ring, then $M(n, R)$ will denote the ring of $n \times n$ matrices with coefficients in R. If A is a symmetric $n \times n$ matrix, and v is an $n \times 1$ column matrix, then $A[v] = {}^{\mathrm{T}}vAv$ is a 1×1 matrix, which we will identify with its scalar value.

We will use the following standard notations for matrices in $GL(2, \mathbf{R})$: E will be the identity matrix,

$$w = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \quad \eta = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix},$$

and if $x \in \mathbf{R}$, we will denote

$$E(x) = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, \quad U_0(x) = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}, \quad U_1(x) = \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}.$$

Let E be the 2×2 identity matrix, and let $J = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$. Then $GSp(4, \mathbf{R})$ is the group of matrices g in $GL(4, \mathbf{R})$ satisfying ${}^{\mathrm{T}}gJg = \mu J$ for some $\mu \in \mathbf{R}$. It is known that if r is even, then all elements of $GSp(2r, \mathbf{R})$ have positive determinant. However, μ may be positive or negative. The subgroup consisting of elements such that $\mu > 0$ will be denoted $GSp^+(4, \mathbf{R})$. The subgroup consisting of elements such that $\mu = 1$ will be denoted $Sp(4, \mathbf{R})$. Thus if $g \in Sp(4, \mathbf{R})$ then ${}^{\mathrm{T}}gJg = J$. We will regard $GL(2, \mathbf{R})$ and its subgroup $SO(2, \mathbf{R})$ as subgroups of $Sp(4, \mathbf{R})$ via the embedding

$$Q \to \begin{pmatrix} Q & \\ & {}^{\mathrm{T}}Q^{-1} \end{pmatrix}.$$

Let $\mathscr{H}_2$ be the Siegel upper half space of genus two. Thus $\mathscr{H}_2$ consists of all 2×2 complex symmetric matrices $Z = X + iY$, such that X and Y are real symmetric matrices with Y positive definite. The group $GSp^+(4, \mathbf{R})$ acts on $\mathscr{H}_2$ in the usual way. Thus if $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GSp^+(4, \mathbf{R})$, where A, B, C and D are 2×2 blocks, then by definition

$$g(Z) = (AZ + B)(CZ + D)^{-1}.$$

If $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GSp^+(4, \mathbf{R})$, let $Z_g = g(iE) = (Ai + B)(Ci + D)^{-1}$. We denote, for

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(4, \mathbf{R}), \quad \lambda, \mu \in \mathbf{C}^2,$$

and for any function ϕ on $Sp(4, \mathbf{R}) \times \mathbf{C}^2$,

$$(\phi \mid \gamma)(g, W) = e^m(-((CZ_g + D)^{-1}C)[W])\phi(\gamma g, {}^{\mathrm{T}}(CZ_g + D)^{-1}W), \quad (1.1)$$

$$\left(\phi \,\|\, \begin{bmatrix} \lambda \\ \mu \end{bmatrix}\right)(g, W) = e^m(Z_g[\lambda] + 2{}^{\mathrm{T}}W\lambda)\phi(g, W + Z_g\lambda + \mu). \quad (1.2)$$

We have the relations

$$\phi \mid \gamma \mid \gamma' = \phi \mid (\gamma\gamma'), \quad (1.3)$$

$$\phi \,\|\, \begin{bmatrix} \lambda \\ \mu \end{bmatrix} \,\|\, \begin{bmatrix} \nu \\ \rho \end{bmatrix} = \phi \,\|\, \begin{bmatrix} \lambda + \nu \\ \mu + \rho \end{bmatrix} \quad \text{provided that } 2m{}^{\mathrm{T}}\lambda\rho \in \mathbf{Z}, \quad (1.4)$$

$$\phi \,\|\, \begin{bmatrix} \lambda \\ \mu \end{bmatrix} \mid \gamma = \phi \mid \gamma \,\|\, \left({}^{\mathrm{T}}\gamma \begin{bmatrix} \lambda \\ \mu \end{bmatrix}\right). \quad (1.5)$$

Let Γ be the subgroup of matrices

$$\left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(4, \mathbf{Z}) \,\middle|\, C \equiv 0 \bmod N, \quad A_{21} \equiv D_{12} \equiv 0 \bmod N \right\}.$$

By a *Jacobi modular form* of level N and index m, we mean a function Φ_0 which is holomorphic in W, and which satisfies

$$\Phi_0 \mid \gamma = \Phi_0 \qquad \text{for } \gamma \in \Gamma; \quad (1.6)$$

$$\Phi_0 \,\|\, \begin{bmatrix} \lambda \\ \mu \end{bmatrix} = \Phi_0 \quad \text{for } \lambda \in \mathbf{Z}^2, \quad \mu \in \frac{1}{N}\mathbf{Z}^2. \quad (1.7)$$

We will also consider the function $\Phi_1 = \Phi_0 \mid J$. It follows that Φ_1 satisfies

$$\Phi_1 \mid \gamma = \Phi_1 \quad \text{for } \gamma \in J^{-1}\Gamma J; \quad (1.8)$$

$$\Phi_1 \,\|\, \begin{bmatrix} \lambda \\ \mu \end{bmatrix} = \Phi_1 \quad \text{for } \lambda \in \frac{1}{N}\mathbf{Z}^2, \quad \mu \in \mathbf{Z}^2. \quad (1.9)$$

We now describe the construction of particular Jacobi modular forms by means of Eisenstein series. Recall that $\Gamma_0(N)$ is the group of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z})$ such that $N \mid c$, and that $\Gamma^0(N)$ is the group of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z})$ such that $N \mid b$. As in the introduction let f be a cuspidal modular form on $\Gamma_0(N)$ of even weight k and trivial character. Thus, we assume that

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau) \quad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N). \quad (1.10)$$

Also let $\hat{f}(\tau) = \tau^{-k} f(-1/N\tau)$. Then $\hat{f}$ also satisfies the same automorphy condition (1.10). It is more convenient to build the Eisenstein series around $\hat{f}$ than the original form f. We will denote by $a(n)$ the Fourier coefficients of f at the cusp at infinity, so that

$$f(\tau) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n\tau}.$$

The L-function of f is then the Dirichlet series $\sum a(n) n^{-s}$.

Let F be the function on $GL^+(2, \mathbf{R})$ given by

$$F(g) = \hat{f}(g(i))(ci + d)^{-k} \det(g)^{k/2} \quad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL^+(2, \mathbf{R}) .$$

It is easy to see that F satisfies

$$F(\gamma g \kappa) = F(g)\rho_k(\kappa) , \tag{1.11}$$

for $\gamma \in \Gamma_0(N)$ and $\kappa \in SO(2, \mathbf{R})$, where

$$\rho_k\left(\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right) = e^{ik\theta} .$$

Let K be the standard maximal compact subgroup $GSp^+(4, \mathbf{R}) \cap O(4)$. Thus K is the stabilizer of iE in the action of $GSp^+(4, \mathbf{R})$ on $\mathscr{H}_2$. Then K is isomorphic to $U(2)$—the isomorphism may be described explicitly as follows. Let $\kappa = A + iB \in U(2)$, where A and B are real matrices. Then $A^{\mathsf{T}}B = B^{\mathsf{T}}A$ and ${}^{\mathsf{T}}AA + {}^{\mathsf{T}}BB = E$, so that the matrix $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ is symplectic, and the homomorphism

$$\kappa \to \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

is an embedding of $U(2)$ into $Sp(4, \mathbf{R})$ which extends the previously mentioned embedding $Q \to \begin{pmatrix} Q & \\ & {}^{\mathsf{T}}Q^{-1} \end{pmatrix}$ of $SO(2)$.

Let $\mathbf{V}$ be a complex vector space, and let $GL(\mathbf{V})$ denote the group of nonsingular linear transformations of $\mathbf{V}$, acting on the *right*. Let $\sigma: K \to GL(\mathbf{V})$ be a finite dimensional complex representation of K which is trivial on $\begin{pmatrix} -E & \\ & -E \end{pmatrix}$, such that there exists a vector $\mathbf{v} \in \mathbf{V}$ with the property that

$$\mathbf{v}\sigma(\kappa) = \rho_k(\kappa) \cdot \mathbf{v} \quad \text{for all } \kappa \in SO(2, \mathbf{R}) . \tag{1.12}$$

It is well known that there are always an infinite number of representations κ having this property.

We will now define a function $I: GSp^+(4, \mathbf{R}) \to \mathbf{V}$ by prescribing that

$$I\left(\begin{pmatrix} Q & X{}^{\mathsf{T}}Q^{-1} \\ & {}^{\mathsf{T}}Q^{-1} \end{pmatrix}\right) = F(Q) \cdot \mathbf{v} \quad \text{for } Q \in GL^+(2, \mathbf{R}), \text{ symmetric } X \in M(2, \mathbf{R}) , \tag{1.13}$$

and

$$I(gz\kappa) = I(g)\sigma(\kappa) \tag{1.14}$$

if z is a scalar matrix and $\kappa \in K$. One may check that there is a unique function I having these properties. Observe that

$$I\left(\begin{pmatrix} U & X{}^{\mathsf{T}}U^{-1} \\ 0 & {}^{\mathsf{T}}U^{-1} \end{pmatrix} g\right) = I(g) \qquad \text{for all } U \in \Gamma_0(N), \text{ symmetric } X \in M(2, \mathbf{R}) \tag{1.15}$$

and

$$I\left(\begin{pmatrix} \eta & \\ & \eta \end{pmatrix}\begin{pmatrix} Q & X\,{}^{\mathsf{T}}Q^{-1} \\ & {}^{\mathsf{T}}Q^{-1} \end{pmatrix}\right) = F(\eta Q\eta)\cdot \mathrm{v}\sigma(\eta) \tag{1.16}$$

for $Q \in GL^+(2, \mathbf{R})$, symmetric $X \in M(2, \mathbf{R})$.

Let s be a complex variable, and for $g \in GSp^+(4, \mathbf{R})$ let

$$I_s(g) = \det(Y_g)^{s/2} I(g)\,. \tag{1.17}$$

Let P be the subgroup of $\begin{pmatrix} Q & X\,{}^{\mathsf{T}}Q^{-1} \\ & {}^{\mathsf{T}}Q^{-1} \end{pmatrix} \in GSp^+(4, \mathbf{R})$ such that $Q \in GL^+(2, \mathbf{R})$ and $X = {}^{\mathsf{T}}X \in M(2, \mathbf{R})$. *Note that Q is assumed to have positive determinant.* Define

$$E_s = \sum_{\substack{\gamma \in P \cap \Gamma \backslash \Gamma \\ \lambda \in \mathbf{Z}^2}} I_s \left\| \begin{bmatrix} \lambda \\ 0 \end{bmatrix} \right| \gamma\,.$$

Thus

$$E_s(g, W) = \sum_{\substack{\lambda \in \mathbf{Z}^2 \\ \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in P \cap \Gamma \backslash \Gamma}} I_s(\gamma g) e^m((\gamma(Z))[\lambda] + 2\,{}^{\mathsf{T}}W(CZ + D)^{-1}\lambda - ((CZ + D)^{-1}C)[W])\,,$$

convergent for $\mathrm{re}(s)$ sufficiently large. Then E_s is a Jacobi modular form of level N.

The group $\Gamma_0(N)$ has $N\prod_{p|N}(1 + p^{-1})$ cusps. These are the points of the rational projective line modulo the action of $\Gamma_0(N)$. Let $\Gamma_0 = \left\{\begin{pmatrix} 1 & \\ n & 1 \end{pmatrix} \middle| n \in \mathbf{Z}\right\}$ be the stabilizer of the cusp 0 in $SL(2, \mathbf{Z})$. Then a complete set of representatives for the set of cusps are the points $\gamma(0)$, where γ runs through a set of coset representatives for $\Gamma_0(N)\backslash SL(2, \mathbf{Z})/\Gamma_0$. We will denote by $a_\gamma(n)$ the Fourier coefficients of $\hat{f}$ at the cusp $\gamma(0)$. Thus if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} w$, let

$$(c\tau + d)^{-k}\hat{f}\left(\frac{a\tau + b}{c\tau + d}\right) = \sum_n a_\gamma(n) e(nN^{-1}\tau)\,,$$

where the summation is over positive $n \in \mathbf{Z}$. In particular, if $\gamma = E$, then $a_\gamma(n)$ equals $a(n)$. We assume that f is normalized so that $a(1) = 1$. We have at each cusp a Fourier expansion

$$F(\gamma w g) = \sum_{0 < n \in \mathbf{Z}} a_\gamma(n) W_n(g)\,,$$

where $W_n(g)$ is the function on $GL^+(2, \mathbf{R})$ defined by

$$W_n\left(\begin{pmatrix} y & x \\ & 1 \end{pmatrix} z\kappa\right) = y^{k/2} e(N^{-1}n(x + iy))\rho_k(\kappa)\,, \tag{1.18}$$

for scalar matrices z, and $\kappa \in SO(2, \mathbf{R})$. Thus

$$N^{-1}\int_{\mathbf{R}/N\mathbf{Z}} F\left(\gamma w \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) e(-N^{-1}nx)\,dx = a_\gamma(n) W_n(g)\,. \tag{1.19}$$

We will express quadratic twists of the L series of f in terms of Fourier-Whittaker coefficients of E_s.

Since f is a newform for $\Gamma_0(M)$, the L-function $L(s,f)=\sum_n a(n)n^{-s}$ has an Euler product of the form

$$L(s,f)=\prod_{p\mid M}(1-a(p)p^{-s})^{-1}\prod_{p\nmid M}(1-a(p)p^{-s}+p^{k-1-2s})^{-1}\,. \tag{1.20}$$

Furthermore, the hypothesis that f is a newform implies that

$$f(-1/N\tau)=\varepsilon(-1)^{k/2}(\sqrt{N}\tau)^k f(\tau)\,, \tag{1.21}$$

where $\varepsilon=\pm 1$. Consequently, if

$$\Lambda(s,f)=N^{s/2}(2\pi)^{-s}\Gamma(s)L(s,f)$$

then

$$\Lambda(s,f)=\varepsilon\Lambda(k-s,f)\,. \tag{1.22}$$

We have

Proposition 1.1. *If p is a prime divisor of M, then*

$$a(p)=\begin{cases}\pm p^{k/2-1} & \text{if } p^2\nmid M;\\ 0 & \text{otherwise.}\end{cases}$$

See Ogg [17], Theorem 1. □

We will denote

$$L_N(s,f)=\prod_{p\nmid N}(1-a(p)p^{-s}+p^{k-1-2s})^{-1}\,.$$

For each p not dividing M, let us factor

$$1-a(p)t+t^2=(1-\sigma_p t)(1-\sigma'_p t)\,,$$

so that

$$L_N(s,f)=\prod_{p\nmid N}(1-\sigma_p p^{-s})^{-1}(1-\sigma'_p p^{-s})^{-1}\,. \tag{1.23}$$

Let

$$L_N(s,f,\vee^2)=\prod_{p\nmid N}(1-\sigma_p^2 p^{-s})^{-1}(1-p^{k-1-s})^{-1}(1-\sigma_p'^2 p^{-s})^{-1}$$

denote the "symmetric square" L-function with p-part removed for all $p\mid N$.

Proposition 1.2. *$L_N(s,f,\vee^2)$ is analytic and nonzero for real $s\geqq k$.*

Proof. This follows from the Rankin-Selberg method. Specifically,

$$L_N(s,f,\vee^2)=\frac{\prod_{p\mid N}L_p(s,f,\vee^2)^{-1}\zeta_M(2s-2k+2)\sum_{n=1}^{\infty}a(n)^2n^{-s}}{\zeta_M(s-k+1)}$$

where

$$L_p(s,f,\vee^2)^{-1}=\begin{cases}1-a(p)p^{-s} & \text{if } p\mid M\,,\\ (1-\sigma_p^2p^{-s})(1-p^{k-1-s})(1-\sigma_p'^2p^{-s}) & \text{if } p\nmid M\,,\end{cases}$$

and $\zeta_M(s)$ is the Riemann zeta function with p-part removed for $p \mid N$. The result follows from the fact that both numerator and denominator have simple poles at $s = k$, and are analytic and strictly positive for real $s > k$. The simple pole of the numerator is proved by Ogg [17], Theorem 3, following Rankin and Selberg. □

Let $L(s, f, \chi_D) = \sum \chi_D(n) a(n) n^{-s}$, and let

$$\Lambda(s, f, \chi_D) = (D^2 M)^{s/2} (2\pi)^{-s} \Gamma(s) L(s, f, \chi_D) .$$

This has analytic continuation as a function of s, and, provided $\gcd(M, D) = 1$ satisfies the functional equation

$$\Lambda(s, f, \chi_D) = \varepsilon \chi_D(-M) \Lambda(k - s, f, \chi_D) . \tag{1.24}$$

This follows from Proposition 3.66 and Lemma 3.63 (2) of Shimura [18].

2. Theta expansions of Jacobi modular forms

In this section, let Φ_0 be a Jacobi modular form of level N, and let $\Phi_1 = \Phi_0 \mid J$. We will obtain expansions of Φ_0 and Φ_1 in terms of theta functions.

The theta functions which we require are defined as follows. Let N_0 be a positive integer, which will eventually be taken to be m/N. Let $Z = X + iY \in \mathscr{H}_2$ and $W \in \mathbf{C}^2$, where X and Y are the real and imaginary parts of Z. If $v \in \mathbf{Z}^2$, then let

$$\theta_v^{N_0}(Z, W) = \sum_{R \equiv v \bmod 2N_0} e\left(\frac{1}{4N_0} Z[R] + {}^{\mathrm{T}}RW\right) . \tag{2.1}$$

The summation is over vectors $R \in \mathbf{Z}^2$ congruent to v modulo $2N_0$. It may be checked that if $\lambda, \rho \in \mathbf{Z}^2$, then

$$\theta(Z, W) = e^{N_0}(Z[\lambda] + 2\,{}^{\mathrm{T}}W\lambda) \theta(Z, W + Z\lambda + \rho) .$$

Consequently if $\mu, v \in \mathbf{Z}^2$, then

$$\theta_v^{N_0}(Z, W) \overline{\theta_\mu^{N_0}(Z, W)} \exp(-4\pi N_0 Y^{-1}[\operatorname{im} W])$$

is invariant under translations of W by the "period lattice" Λ_Z consisting of vectors $Z\lambda + \rho$ such that $\lambda, \rho \in \mathbf{Z}^2$.

Proposition 2.1. *We have*

$$\int_{C^2/\Lambda_Z} \theta_v^{N_0}(Z, W) \overline{\theta_\mu^{N_0}(Z, W)} \exp(-4\pi N_0 Y^{-1}[\operatorname{im} W]) \, dW = \begin{cases} \dfrac{\sqrt{\det Y}}{2N_0} & \text{if } \mu \equiv v \bmod 2N_0 \,; \\ 0 & \text{otherwise} \,. \end{cases}$$

Here dW denotes Lebesgue measure. We omit the proof, which is similar to Theorem 5.3 of Eichler and Zagier [5]. □

It follows from the periodicity properties implied by (1.6–9) that Φ_j, for $j = 0$ or 1, has a Fourier expansion

$$\Phi_j(g, W) = \sum_{T, R} B_j(g; T, R) e(N^{-j} \operatorname{tr}(TZ_g)) e(N^{1-j}\,{}^{\mathrm{T}}RW) , \tag{2.2}$$

where the summation runs over 2×2 half-integral symmetric matrices T, and integer column vectors R, and where

$B_j(g; T, R) =$

$$N^{-3j} \int_{(\mathbf{R}/N^j\mathbf{Z})^3} \int_{(\mathbf{R}/\mathbf{Z})^2} \Phi_j\left(\begin{pmatrix} E & X \\ & E \end{pmatrix} g, W\right) e(-N^{-j}\operatorname{tr}(T(Z_g + X)) - N^{1-j}\,{}^{\mathsf{T}}RW)\,dW\,dX. \tag{2.3}$$

We are identifying a symmetric matrix $X = \begin{pmatrix} x_4 & x_3 \\ x_3 & x_1 \end{pmatrix}$ with the element (x_1, x_3, x_4) of $\mathbf{R}^3$.

Proposition 2.2. *Let $j = 0$ or 1, and let $v \in \mathbf{Z}^2$. Then there exist functions $\mathscr{E}_j(g; v)$ of $g \in \mathscr{H}_2$ such that*

$$\Phi_j(g, W) = \sum_{v \bmod 2m/N} \mathscr{E}_j(g; v)\theta_v^{m/N}(N^{1-2j}Z_g, N^{1-j}W). \tag{2.4}$$

We have a Fourier expansion

$$\mathscr{E}_j(g; v) = \sum_U C_j(g; U, v)\, e\left(\frac{1}{4mN^j}\operatorname{tr}(UZ_g)\right), \tag{2.5}$$

where the summation is over integral symmetric matrices U, and

$$C_j(g; U, v) = \begin{cases} B_j(g; T, v) & \text{if } T = \dfrac{1}{4m}(U + N^{2-j}v\,{}^{\mathsf{T}}v) \text{ is half-integral}; \\ 0 & \text{otherwise}. \end{cases} \tag{2.6}$$

We have the transformation properties

$$\mathscr{E}_1(g; v) = \frac{\sqrt{-\det Z_g}}{2m} \sum_{\mu \bmod 2m/N} e\left(-\frac{N}{2m}{}^{\mathsf{T}}v\mu\right)\mathscr{E}_0(Jg; \mu) \tag{2.7}$$

and

$$\mathscr{E}_0(g; v) = \frac{\sqrt{-\det Z_g}}{2m} \sum_{\mu \bmod 2m/N} e\left(\frac{N}{2m}{}^{\mathsf{T}}v\mu\right)\mathscr{E}_1(Jg; \mu). \tag{2.8}$$

Remark. Here, as in other places throughout this paper, the square root must be taken in the obvious way—in this case, $\sqrt{-\det Z_g}$ is chosen positive when Z_g is pure imaginary.

Proof. We will prove first that if $\lambda \in N^{-j}\mathbf{Z}^2$, then

$$B_j(g; T, R) = B_j\left(g; T - \frac{N}{2}R\,{}^{\mathsf{T}}\lambda - \frac{N}{2}\lambda\,{}^{\mathsf{T}}R + mN^j\lambda\,{}^{\mathsf{T}}\lambda, R - 2mN^{j-1}\lambda\right). \tag{2.9}$$

Observe that by (1.7) and (1.9), we have

$$\Phi_j(g, W) = e(mZ_g[\lambda] + 2m\,{}^{\mathsf{T}}\lambda W)\Phi_j(g, W + Z_g\lambda) \qquad \text{for } \lambda \in N^{-j}\mathbf{Z}^2.$$

Substitute

$$\begin{pmatrix} E & X \\ & E \end{pmatrix} g$$

for g in this relation and substitute into the definition (2.3) of $B_j(g; T, R)$. We obtain for $B_j(g; T, R)$ the expression

$$N^{-3j} \int_{(\mathbf{R}/N^j\mathbf{Z})^3} \int_{(\mathbf{R}/\mathbf{Z})^2} \Phi_j\left(\begin{pmatrix} E & X \\ & E \end{pmatrix} g, W + (Z_g + X)\lambda\right) \times$$
$$e(m(Z_g + X)[\lambda] + 2m{}^{\mathrm{T}}\lambda W - N^{-j}\mathrm{tr}(T(Z_g + X)) - N^{1-j}{}^{\mathrm{T}}RW)\,dW\,dX\ .$$

Replace W by $W - (Z_g + X)\lambda$ in this integral. This becomes

$$N^{-3j} \int_{(\mathbf{R}/N^j\mathbf{Z})^3} \int_{(\mathbf{R}/\mathbf{Z})^2} \Phi_j\left(\begin{pmatrix} E & X \\ & E \end{pmatrix} g, W\right) e(-m(Z_g + X)[\lambda] +$$
$$N^{1-j}{}^{\mathrm{T}}R(Z_g + X)\lambda - N^{-j}\mathrm{tr}(T(Z_g + X)) - N^{1-j}({}^{\mathrm{T}}R - 2mN^{j-1}{}^{\mathrm{T}}\lambda)W)\,dW\,dX\ .$$

(Note that we have used the fact that Φ_j is holomorphic as a function of W to justify moving the path of integration.) Now observing that

$$\mathrm{tr}(\lambda{}^{\mathrm{T}}\lambda(Z_g + X)) = (Z_g + X)[\lambda]\,, \quad \mathrm{tr}((\tfrac{1}{2}R{}^{\mathrm{T}}\lambda + \tfrac{1}{2}\lambda{}^{\mathrm{T}}R)(Z_g + X)) = {}^{\mathrm{T}}R(Z_g + X)\lambda\,,$$

we see that this expression for $B_j(g; T, R)$ is essentially the same as the definition, by equation (2.3), of $B_j\left(g; T - \frac{N}{2}R{}^{\mathrm{T}}\lambda - \frac{N}{2}\lambda{}^{\mathrm{T}}R + mN^j\lambda{}^{\mathrm{T}}\lambda, R - 2mN^{j-1}\lambda\right)$, whence (2.9).

It is easy to see that given T, T', R, R' the two equations

$$T' = T - \frac{N}{2}R{}^{\mathrm{T}}\lambda - \frac{N}{2}\lambda{}^{\mathrm{T}}R + mN^j\lambda{}^{\mathrm{T}}\lambda\,,$$
$$R' = R - 2mN^{j-1}\lambda$$

may be solved (for $\lambda \in N^{-j}\mathbf{Z}^2$) if and only if

$$R' \equiv R \bmod 2m/N\ ,$$

and

$$4mT' - N^{2-j}R'{}^{\mathrm{T}}R' = 4mT - N^{2-j}R{}^{\mathrm{T}}R\ .$$

Thus we introduce the notation, for integral matrices U,

$$C_j(g; U, v) = \begin{cases} B_j(g; T, R) & \text{if there exist } R \equiv v \bmod 2m/N, \text{ half integral } T \\ & \text{such that } U = 4mT - N^{2-j}R{}^{\mathrm{T}}R\ ; \\ 0 & \text{otherwise}\ . \end{cases} \qquad (2.10)$$

The substance of (2.9) is that this is well-defined. Note that if there exists any integral $R \equiv v \bmod 2m/N$ and half integral T such that $U = 4mT - N^{2-j}R{}^{\mathrm{T}}R$, then $\frac{1}{4m}(U + N^{2-j}v{}^{\mathrm{T}}v)$ is necessarily half integral, so that we may take $R = v$ and $T = \frac{1}{4m}(U + N^{2-j}v{}^{\mathrm{T}}v)$ in defining U. Thus the definition (2.6) of $C_j(g; U, v)$ is equivalent to the definition (2.10).

We now have

$$\Phi_j(g, W) = \sum_{\nu \bmod 2m/N} \sum_U C_j(g; U, \nu) \sum_{R \equiv \nu \bmod 2m/N} e\left(\frac{1}{4mN^j} \operatorname{tr}((U + N^{2-j} R^{\mathsf T} R) Z_g)\right) e(N^{1-j\,\mathsf T} R W), \tag{2.11}$$

whence (2.4), where $\mathscr{E}_j(g; \nu)$ is defined by (2.5). It remains for us to prove the transformation properties (2.7) and (2.8). We will prove (2.7)—the proof of (2.8) is similar (or (2.8) may be deduced from (2.7)).

It follows from the Poisson summation formula that

$$\theta_\nu^{N_0}(-Z^{-1}, Z^{-1} W) = e(N_0 Z^{-1}[W]) \frac{\sqrt{-\det Z}}{2N_0} \sum_{\mu \bmod 2N_0} e\left(-\frac{1}{2N_0} {}^{\mathsf T}\nu\mu\right) \theta_\mu^{N_0}(Z, W), \tag{2.12}$$

or equivalently

$$\theta_\mu^{N_0}(Z, W) = e(-N_0 Z^{-1}[W]) \frac{1}{2N_0 \sqrt{-\det Z}} \sum_{\mu \bmod 2N_0} e\left(-\frac{1}{2N_0} {}^{\mathsf T}\nu\mu\right) \theta_\nu^{N_0}(-Z^{-1}, -Z^{-1} W). \tag{2.13}$$

On the other hand, we have by definition

$$\Phi_1(g, W) = e(-m Z_g^{-1}[W]) \Phi_0(Jg, -Z_g^{-1} W). \tag{2.14}$$

Substituting (2.5) for Φ_j on both sides of (2.14) and applying (2.12), we obtain

$$\sum_{\nu \bmod 2m/N} \mathscr{E}_1(g; \nu) \theta_\nu^{m/N}(N^{-1} Z_q, W) = \sum_{\nu \bmod 2m/N} \frac{\sqrt{-\det Z_g}}{2m} \sum_{\mu \bmod 2m/N} e\left(-\frac{N}{2m} {}^{\mathsf T}\nu\mu\right) \mathscr{E}_0(Jg; \mu) \theta_\nu^{m/N}(N^{-1} Z_g, W).$$

Now (2.7) follows from Proposition 2.1. This completes the proof of Proposition 2.2. □

Proposition 2.3. *Suppose that* $\nu \in \mathbf{Z}^2$. *Then if* $n \in \mathbf{Z}$, $N \mid n$, *we have*

$$\mathscr{E}_1\left(\begin{pmatrix} E(n) & \\ & {}^{\mathsf T}E(n)^{-1} \end{pmatrix} g; \nu\right) = \mathscr{E}_1(g; {}^{\mathsf T}E(n)\nu), \tag{2.15}$$

$$\mathscr{E}_0\left(\begin{pmatrix} {}^{\mathsf T}E(n) & \\ & E(n)^{-1} \end{pmatrix} g; \mu\right) = \mathscr{E}_0(g; E(n)\mu). \tag{2.16}$$

Proof. By (1.8),

$$\Phi_1(g, W) = \Phi_1\left(\begin{pmatrix} E(n) & \\ & {}^{\mathsf T}E(n)^{-1} \end{pmatrix} g, E(n) W\right).$$

By (2.4), the left side of this equation equals

$$\sum_{\nu \bmod 2m/N} \mathscr{E}_1(g;\nu)\theta_\nu^{m/N}(N^{-1}Z_g, W)\,.$$

The right side equals

$$\sum_{\nu \bmod 2m/N} \mathscr{E}_1\left(\begin{pmatrix} E(n) & \\ & {}^{\mathsf{T}}E(n)^{-1}\end{pmatrix} g;\nu\right)\theta_\nu^{m/N}(N^{-1}E(n)Z_g{}^{\mathsf{T}}E(n), E(n)W)\,.$$

Now

$$\theta_\nu^{m/N}(N^{-1}E(n)Z_g{}^{\mathsf{T}}E(n), E(n)W) = \theta_{{}^{\mathsf{T}}E(n)\nu}^{m/N}(N^{-1}Z_g, W)\,.$$

Now (2.15) follows from the linear independence of the theta functions $\theta_\nu^{m/N}$ for ν modulo $2m/N$, which is a consequence of the orthogonality relation of Proposition 2.1.

The proof of (2.16) is similar. □

Corollary 2.4. *Suppose that ν has the special form ${}^{\mathsf{T}}(0, \nu_2)$. If $n \in \mathbf{Z}$ and $N \mid n$ then*

$$\mathscr{E}_1\left(\begin{pmatrix} E(n) & \\ & {}^{\mathsf{T}}E(n)^{-1}\end{pmatrix} g;\nu\right) = \mathscr{E}_1(g;\nu)\,, \tag{2.17}$$

$$\sum_{\mu \bmod 2m/N} e\left(-\frac{N}{2m}{}^{\mathsf{T}}\nu\mu\right)\mathscr{E}_0\left(\begin{pmatrix} {}^{\mathsf{T}}E(n) & \\ & E(n)^{-1}\end{pmatrix} g;\mu\right) = \sum_{\mu \bmod 2m/N} e\left(-\frac{N}{2m}{}^{\mathsf{T}}\nu\mu\right)\mathscr{E}_0(g;\mu)\,. \tag{2.18}$$

Proof. This is an immediate consequence of Proposition 2.3. □

Proposition 2.5. *Suppose that $\nu, \mu \in \mathbf{Z}^2$, and suppose that ν has the special form ${}^{\mathsf{T}}(0, \nu_2)$. If $V = (V_{ij})$ is an integral symmetric matrix such that $V_{11} \equiv V_{12} \equiv 0 \bmod N$ and $V_{22} \equiv 0 \bmod 4m$, then*

$$\mathscr{E}_1\left(\begin{pmatrix} 1 & V \\ 0 & 1\end{pmatrix} g;\nu\right) = \mathscr{E}_1(g;\nu) \tag{2.19}$$

If V' is an arbitrary integral symmetric matrix, then

$$\mathscr{E}_0\left(\begin{pmatrix} 1 & V' \\ 0 & 1\end{pmatrix} g;\mu\right) = \mathscr{E}_0(g;\mu)\,. \tag{2.20}$$

Proof. By (2.6), given the special form of ν, we have $C_1(g; U, \nu) = 0$ unless the integral symmetric matrix $U = (U_{ij})$ satisfies $U_{11} \equiv 2U_{12} \equiv 0 \bmod 4m$, and $U_{22} \equiv 0 \bmod N$. This implies (2.19). On the other hand, since $m \mid N^2$, $C_0(g; U, \mu) = 0$ unless $U_{11} \equiv U_{12} \equiv U_{22} \equiv 0 \bmod m$. Since we have in fact $4m \mid N^2$, this implies (2.20). □

Proposition 2.6. *Suppose that $\nu \in \mathbf{Z}^2$ has the special form ${}^{\mathsf{T}}(0, \nu_2)$. Then if $n \in \mathbf{Z}$,*

$$\sqrt{\det(E + U_1(n)Z_g)}\,\mathscr{E}_1\left(\begin{pmatrix} E & \\ U_1(n) & E\end{pmatrix} g;\nu\right) = \mathscr{E}_1(g;\nu)\,. \tag{2.21}$$

Furthermore, if $N \mid n$ then

$$\sqrt{\det(E+U_0(n)Z_g)}\sum_{\mu \bmod 2m/N} e\left(-\frac{N}{2m}{}^{\mathsf T}\nu\mu\right)\mathscr{E}_0\left(\begin{pmatrix} E & \\ U_0(n) & E\end{pmatrix}g;\nu\right)$$

$$=\sum_{\mu \bmod 2m/N} e\left(-\frac{N}{2m}{}^{\mathsf T}\nu\mu\right)\mathscr{E}_0(g;\nu)\,, \quad (2.22)$$

Proof. If $g_1=\begin{pmatrix} E & \\ U_1(x) & E\end{pmatrix}g$, then $Z_{g_1}=Z_g(E+U_1(x)Z_g)^{-1}$. Thus by (2.7),

$$\sqrt{\det(E+U_1(x)Z_g)}\,\mathscr{E}_1\left(\begin{pmatrix} E & \\ U_1(x) & E\end{pmatrix}g;\nu\right)$$

$$=\frac{\sqrt{-\det Z_g}}{2m}\sum_{\mu \bmod 2m/N} e\left(-\frac{N}{2m}{}^{\mathsf T}\nu\mu\right)\mathscr{E}_0\left(\begin{pmatrix} E & U_1(-x)\\ & E\end{pmatrix}Jg;\mu\right).$$

(2.21) follows from this expression and (2.20).

On the other hand, if $g_0=\begin{pmatrix} E & \\ U_0(x) & E\end{pmatrix}g$, then $Z_{g_0}=Z_g(E+U_0(x)Z_g)^{-1}$, and so by (2.8)

$$\sqrt{\det(E+U_0(x)Z_g)}\sum_{\mu \bmod 2m/N} e\left(-\frac{N}{2m}{}^{\mathsf T}\nu\mu\right)\mathscr{E}_0\left(\begin{pmatrix} E & \\ U_0(x) & E\end{pmatrix}g;\nu\right)$$

$$=\frac{\sqrt{-\det Z_g}}{2m}\sum_{\mu,\,\rho \bmod 2m/N} e\left(\frac{N}{2m}{}^{\mathsf T}(\rho-\nu)\mu\right)\mathscr{E}_1\left(\begin{pmatrix} E & U_0(-x)\\ & E\end{pmatrix}Jg;\rho\right).$$

For fixed ρ, the sum over μ vanishes unless $\rho\equiv\nu \bmod 2m/N$, so this equals

$$\frac{2m\sqrt{-\det Z_g}}{N^2}\mathscr{E}_1\left(\begin{pmatrix} E & U_0(-x)\\ & E\end{pmatrix}Jg;\nu\right).$$

Now (2.22) follows from this expression and (2.19). □

Proposition 2.7. *If $j=1$ and $\gamma\in\Gamma^0(N)$, or if $j=0$ and $\gamma\in\Gamma_0(N)$, then*

$$B_j\left(\begin{pmatrix}\gamma & \\ & {}^{\mathsf T}\gamma^{-1}\end{pmatrix}g;T,R\right)=B_j(g;{}^{\mathsf T}\gamma T\gamma,{}^{\mathsf T}\gamma R)\,, \quad (2.23)$$

$$C_j\left(\begin{pmatrix}\gamma & \\ & {}^{\mathsf T}\gamma^{-1}\end{pmatrix}g;U,R\right)=C_j(g;{}^{\mathsf T}\gamma U\gamma,{}^{\mathsf T}\gamma R)\,, \quad (2.24)$$

Proof. By (2.3), the left side of (2.23) equals

$$N^{-3j}\int_{(\mathbf{R}/N^j\mathbf{Z})^3}\int_{(\mathbf{R}/\mathbf{Z})^2}\Phi_j\left(\begin{pmatrix}E & X\\ & E\end{pmatrix}\begin{pmatrix}\gamma & \\ & {}^{\mathsf T}\gamma^{-1}\end{pmatrix}g,W\right)\times$$

$$e(-N^{-j}\operatorname{tr}(T(\gamma Z_g{}^{\mathsf T}\gamma+X))-N^{1-j}\,{}^{\mathsf T}RW)\,dW\,dX\,.$$

In this formula we substitute $\gamma X{}^{\mathsf T}\gamma$ for X, and γW for W. Since

$$\Phi_j\left|\begin{pmatrix}\gamma & \\ & {}^{\mathsf T}\gamma^{-1}\end{pmatrix}\right.=\Phi_j\,,$$

this equals

$$N^{-3j} \int_{(\mathbf{R}/N^j\mathbf{Z})^3} \int_{(\mathbf{R}/\mathbf{Z})^2} \Phi_j\left(\begin{pmatrix} E & X \\ & E \end{pmatrix} g, W\right) \times$$

$$e(-N^{-j}\operatorname{tr}({}^{\mathsf{T}}\gamma T \gamma (Z_g + X)) - N^{1-j}\,{}^{\mathsf{T}}({}^{\mathsf{T}}\gamma R) W)\, dW\, dX\,,$$

whence (2.23). Equation (2.24) is a consequence of (2.23). □

Corollary 2.8. *Suppose that v has the special form ${}^{\mathsf{T}}(0, v_2)$, and that $n, D \in \mathbf{Z}$, $N \mid n$. Then*

$$C_1\left(\begin{pmatrix} E(n) & \\ & {}^{\mathsf{T}}E(n)^{-1} \end{pmatrix} g;\, U_1(ND), v\right) = C_1(g;\, U_1(ND), v)\,, \tag{2.25}$$

$$\sum_{\mu \bmod 2m/N} e\left(\frac{N}{2m}\, {}^{\mathsf{T}}v\mu\right) C_0\left(\begin{pmatrix} {}^{\mathsf{T}}E(n)^{-1} & \\ & E(n) \end{pmatrix} g;\, U_0(D), \mu\right)$$

$$= \sum_{\mu \bmod 2m/N} e\left(\frac{N}{2m}\, {}^{\mathsf{T}}v\mu\right) C_0(g;\, U_0(D), \mu)\,. \tag{2.26}$$

Proof. This follows from Proposition 2.7 and the fact that ${}^{\mathsf{T}}E(n)v = v$, together with ${}^{\mathsf{T}}E(n)U_1(ND)E(n) = U_1(ND)$ and $E(n)U_0(D){}^{\mathsf{T}}E(n) = U_0(D)$. □

3. Whittaker functions on the metaplectic group

Let us define two *nondegenerate Whittaker functions* W^+ and W^- which are associated with Eisenstein series of half-integral weight on $GSp(4, \mathbf{R})$. Let $\sigma: K \to GL(\mathbf{V})$ be a representation of K, and let $\mathbf{v} \in \mathbf{V}$ be a vector. Let s be a complex number which is initially assumed to have real part greater than 2. Then, if $y_1, y_2 > 0$, we define

$$W^\pm(y_1, y_2; s) = W^\pm(y_1, y_2; s; \mathbf{v}, \sigma) =$$

$$(y_1 y_2)^{4-s} y_2^{k/2} \int_{\mathbf{R}^3} \frac{\sqrt{-\det(X + iE)}}{|\det(X + iE)|^s} e(\pm y_1 x_1) e(y_2(x_2' + iy_2'))(y_2')^{k/2} \mathbf{v}\sigma(\kappa)\, dX \tag{3.1}$$

provided that $\mathbf{v}$ satisfies (1.12). In this formula, the notation is as follows. We are identifying a real symmetric matrix $X = \begin{pmatrix} x_4 & x_3 \\ x_3 & x_1 \end{pmatrix}$ with the point (x_1, x_3, x_4) of $\mathbf{R}^3$, x_2', y_2' and $\kappa = \kappa(X)$ are described as follows. Let $w = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$ and determine Q' of the form $Q' = \sqrt{y_1'}\begin{pmatrix} y_2' & x_2' \\ & 1 \end{pmatrix}$ and κ a unitary similitude (which depends on X) so that

$$\begin{pmatrix} & w \\ -w & \end{pmatrix}\begin{pmatrix} E & X \\ & E \end{pmatrix} = \begin{pmatrix} Q' & X'{}^{\mathsf{T}}Q'^{-1} \\ & {}^{\mathsf{T}}Q'^{-1} \end{pmatrix}\kappa\,. \tag{3.2}$$

According to the Iwasawa decomposition, Q' and κ are uniquely determined. Also, the branch of $\sqrt{-\det(X + iE)}$ is chosen to be positive when $X = 0$, and extended to all real symmetric matrices X by continuity. $\sigma: K \to GL(\mathbf{V})$ is a representation and $\mathbf{v} \in \mathbf{V}$ is a vector satisfying (1.12). We will prove shortly that the integral (3.1) is absolutely convergent if $\mathrm{re}(s) > 2$.

We may compute κ, x'_2 and y'_2 as follows. Let us denote

$$\kappa^{-1} = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}, \tag{3.3}$$

so that $A + Bi$ is unitary, ${}^{\mathrm{T}}AB = {}^{\mathrm{T}}BA$ and ${}^{\mathrm{T}}AA + {}^{\mathrm{T}}BB = E$. Thus $A + XB = 0$, so that B must be nonsingular and $X = -AB^{-1}$. These may be computed as follows. We have, since X is symmetric, $X^2 = {}^{\mathrm{T}}B^{-1}\,{}^{\mathrm{T}}AAB^{-1} = {}^{\mathrm{T}}B^{-1}(E - {}^{\mathrm{T}}BB)B^{-1} = {}^{\mathrm{T}}B^{-1}B^{-1} - E$. Therefore

$$B^{\mathrm{T}}B = (X^2 + E)^{-1}\,. \tag{3.4}$$

Note that $wB = Q'$. Thus x'_2, y'_2 may be computed by comparing coefficients in (3.4). The most efficient way to do this is to introduce new variables as follows. It follows from the principal axis theorem that we have a factorization:

$$X = \begin{pmatrix} x_4 & x_3 \\ x_3 & x_1 \end{pmatrix} = \begin{pmatrix} c & d \\ -d & c \end{pmatrix}\begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}\begin{pmatrix} c & -d \\ d & c \end{pmatrix}, \tag{3.5}$$

where $c^2 + d^2 = 1$. Then (3.4) implies that

$$x'_2 = \frac{cd(\alpha^2 - \beta^2)}{1 + c^2\beta^2 + d^2\alpha^2}, \qquad y'_2 = \frac{\sqrt{(1 + \alpha^2)(1 + \beta^2)}}{1 + c^2\beta^2 + d^2\alpha^2}\,. \tag{3.6}$$

We have $\det(X + iE) = (\alpha + i)(\beta + i)$. Furthermore, κ is now a function of X which may be described as follows. We have

$$\begin{aligned} A &= \begin{pmatrix} c & d \\ -d & c \end{pmatrix}\begin{pmatrix} -\frac{\alpha}{\sqrt{1+\alpha^2}} & \\ & -\frac{\beta}{\sqrt{1+\beta^2}} \end{pmatrix}\begin{pmatrix} c & -d \\ d & c \end{pmatrix}U\,, \\ B &= \begin{pmatrix} c & d \\ -d & c \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{1+\alpha^2}} & \\ & \frac{1}{\sqrt{1+\beta^2}} \end{pmatrix}\begin{pmatrix} c & -d \\ d & c \end{pmatrix}U\,, \end{aligned} \tag{3.7}$$

where U is an orthogonal matrix of determinant one which is determined by the requirement that the upper left hand corner entry of B is zero, and that the upper right corner entry of B is positive (so that $wB = Q'$ will have the required form).

We may translate (3.6) into the following more explicit formulas:

$$x'_2 = \frac{-x_3(x_1 + x_4)}{1 + x_1^2 + x_3^2}, \qquad y'_2 = \frac{\sqrt{1 + x_1^2 + 2x_3^2 + x_4^2 + (x_1x_4 - x_3^2)^2}}{1 + x_1^2 + x_3^2}\,. \tag{3.8}$$

Proposition 3.1. *The integral*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (1 + x_1^2 + 2x_3^2 + x_4^2 + (x_1x_4 - x_3^2)^2)^{-\alpha} (1 + x_1^2 + x_3^2)^{-\beta}(1 + x_1^2)^{-\gamma}\,dx_1\,dx_3\,dx_4 \tag{3.9}$$

is absolutely convergent provided that

$$\alpha > \tfrac{1}{2}, \qquad 2\alpha + \beta > \tfrac{3}{2}, \qquad \alpha + \beta + \gamma > 1 .$$

Proof. The integrals with respect to x_4, x_3 and x_1 may be carried out (in that order) by use of the formula

$$\int_{-\infty}^{\infty} (Ax^2 + Bx + C)^{-\nu}\, dx = \sqrt{\pi}\frac{\Gamma(\nu - \frac{1}{2})}{\Gamma(\nu)} 2^{2\nu - 1}(4AC - B^2)^{-\nu + 1/2} A^{\nu - 1} , \qquad (3.10)$$

valid if $4AC - B^2 > 0$ and $\mathrm{re}(\nu) > 1/2$. The three successive integrations requires $\alpha > \frac{1}{2}$, $2\alpha + \beta > \frac{3}{2}$, and $\alpha + \beta + \gamma > 1$. □

Proposition 3.2. *The integral* (3.1) *is absolutely convergent if* $\mathrm{re}(s) > 2$.

Proof. Since $k \geqq 2$, we may estimate

$$e(y_2(x_2' + iy_2'))(y_2')^{k/2} \ll y_2'^{1/2} .$$

Thus the integrand is dominated by

$$\left|\frac{\sqrt{-\det(X + iE)}}{|\det(X + iE)|^s} y_2'\right| = (1 + x_1^2 + 2x_3^2 + x_4^2 + (x_1 x_4 - x_3)^2)^{(1-s)/2}(1 + x_1^2 + x_3^2)^{-1/2} .$$

The result now follows from Proposition 3.1. □

Although we have the convergence of the integral for $\mathrm{re}(s) > 2$, we will be particularly concerned with the behavior of the Whittaker function at $s = 2$, and so we need to know that the Whittaker function has analytic continuation to the left. Obtaining this analytic continuation will be our next objective. The proof of this is influenced by the thesis of Jacquet [9].

Let us introduce a variation of this Whittaker function which contains a second independent variable r. Let us define a function $\mathscr{I}_{\mathbf{v}}(g)$ on $GSp^+(4, \mathbf{R})$ by

$$\mathscr{I}_{\mathbf{v}}\left(\begin{pmatrix} E(x_2) & \\ & {}^{\mathrm{T}}E(-x_2)\end{pmatrix}\begin{pmatrix} Y & X^{\mathrm{T}}Y^{-1} \\ & {}^{\mathrm{T}}Y^{-1}\end{pmatrix}\kappa\right) = |\det Y|^s y_2^r \mathbf{v}\sigma(\kappa) . \qquad (3.11)$$

where κ is a unitary similitude, and $Y = \sqrt{y_1}\begin{pmatrix} y_2 & \\ & 1\end{pmatrix}$. (This depends on r and s, but we suppress this dependence from the notation.) Define

$$V^{\pm}(y_1, y_2; s, r) = V^{\pm}(y_1, y_2; s, r; \mathbf{v}, \sigma) =$$

$$(-1)^{k/2} y_1^{4-s} y_2^{5-r-s} \int_{\mathbf{R}}\int_{\mathbf{R}^3} \mathscr{I}_{\mathbf{v}}\left(J\begin{pmatrix} E(x_2) & \\ & {}^{\mathrm{T}}E(-x_2)\end{pmatrix}\begin{pmatrix} E & X \\ & E\end{pmatrix}\right)$$

$$\times \sqrt{-\det(X + iE)}\, e(\pm y_1 x_1) e(y_2 x_2)\, dX\, dx_2 . \qquad (3.12)$$

It may be checked that this integral is absolutely convergent if $\mathrm{re}(r) > 1/2$ and $\mathrm{re}(s - r) > 3/2$. In this definition we do *not* necessarily require that $\mathbf{v}$ satisfies (1.12). If it does, however, we denote

$$W^{\pm}(y_1, y_2; s, r) = W^{\pm}(y_1, y_2; s, r; \mathbf{v}, \sigma) = \pi^{-r}\Gamma\left(r + \frac{k}{2}\right) V^{\pm}(y_1, y_2; s, r; \mathbf{v}, \sigma) .$$

Proposition 3.3. (*i*). *The integral* (3.12), *initially defined for* $\mathrm{re}(r) > 1/2$ *and* $\mathrm{re}(s-r) > 3/2$, *has analytic continuation to the region* $\mathrm{re}(s+r) > 5/2$ *and* $\mathrm{re}(s-r) > 3/2$.

(*ii*). *If* $\mathbf{v}$ *satisfies* (1.12), *we have the functional equation*

$$W^{\pm}(y_1, y_2; s, r) = W^{\pm}(y_1, y_2; s, 1-r)\,. \tag{3.13}$$

If $\mathrm{re}(s) > (3+k)/2$, *then*

$$W^{\pm}\left(y_1, y_2; s, \frac{k}{2}\right) = W^{\pm}(y_1, y_2; s)\,. \tag{3.14}$$

Proof. For an arbitrary vector $\mathbf{v}$, we may decompose $\mathbf{v}$ into a sum of vectors satisfying (1.12) for various values of k. Thus in order to prove the analytic continuation of (3.12), there is no harm in assuming (1.12). Also, since the Gamma function has no zeros, the analytic continuation of $V^{\pm}$ will follow if we prove the analytic continuation of $W^{\pm}$.

Let Q', κ be as in (3.2), and let $Q' = E(x_2')Y'$, where $Y' = \sqrt{y_1'}\begin{pmatrix} y_2' & \\ & 1 \end{pmatrix}$. Note

$$J\begin{pmatrix} E(x_2) & \\ & {}^{\top}E(-x_2) \end{pmatrix} = \begin{pmatrix} w & \\ & w \end{pmatrix}\begin{pmatrix} E(x_2) & \\ & {}^{\top}E(-x_2) \end{pmatrix}\begin{pmatrix} & w \\ -w & \end{pmatrix}.$$

Making the substitution $x_2 \to x_2 - x_2'$, and using the independence of (3.11) on X, we obtain

$$W^{\pm}(y_1, y_2; s, r) =$$

$$(-1)^{k/2}\pi^{-r}\Gamma\left(r+\frac{k}{2}\right)y_1^{4-s}y_2^{5-r-s}\int_{\mathbf{R}}\int_{\mathbf{R}^3}\mathscr{I}_{\mathbf{v}}\left(\begin{pmatrix} w & \\ & w \end{pmatrix}\begin{pmatrix} E(x_2) & \\ & {}^{\top}E(-x_2) \end{pmatrix}\times\right.$$
$$\left.\begin{pmatrix} Y' & \\ & {}^{\top}Y'^{-1} \end{pmatrix}\kappa\right)\sqrt{-\det(X+iE)}\,e(\pm y_1x_1)e(y_2x_2')e(-y_2x_2)\,dX\,dx_2\,.$$

Let $\delta = \sqrt{x_2^2 + y_2'^2}$ and

$$Q_1 = \sqrt{\delta^2 y_1'}\begin{pmatrix} y_2'\delta^{-2} & -x_2\delta^{-2} \\ & 1 \end{pmatrix}.$$

Let

$$\kappa' = \begin{pmatrix} x_2\delta^{-1} & -y_2'\delta^{-1} \\ y_2'\delta^{-1} & x_2\delta^{-1} \end{pmatrix}.$$

As usual, we identify this matrix in $SO(2)$ with its image in K under the embedding $\kappa \to \begin{pmatrix} \kappa & \\ & {}^{\top}\kappa^{-1} \end{pmatrix}$. Now

$$\begin{pmatrix} w & \\ & w \end{pmatrix}\begin{pmatrix} E(x_2) & \\ & {}^{\top}E(-x_2) \end{pmatrix}\begin{pmatrix} Y' & \\ & {}^{\top}Y'^{-1} \end{pmatrix}\kappa = \begin{pmatrix} Q_1 & \\ & {}^{\top}Q_1^{-1} \end{pmatrix}\kappa'\kappa\,.$$

Furthermore, by (1.12) we have

$$\mathbf{v}\sigma(\kappa') = \left(\frac{x_2 - iy_2'}{x_2 + iy_2'}\right)^{k/2} \mathbf{v} .$$

Thus

$$W^{\pm}(y_1, y_2; s, r) =$$

$$(-1)^{k/2}\pi^{-r}\Gamma\left(r+\frac{k}{2}\right)y_1^{4-s}y_2^{5-r-s}\int_{\mathbf{R}}\int_{\mathbf{R}^3}(y_1'y_2')^s\delta^{-2r}y_2'^r\left(\frac{x_2 - iy_2'}{x_2 + iy_2'}\right)^{k/2}$$

$$\times\sqrt{-\det(X+iE)}\,e(\pm y_1x_1)e(y_2x_2')e(-y_2x_2)\mathbf{v}\sigma(\kappa)\,dX\,dx_2 .$$

It follows from (3.4) that $y_1'y_2' = \det(Q') = \det(B) = |\det(X + iE)|^{-1}$. Moreover, according to Gradshteyn and Ryzhik [8] (3.384.9), we have the identity

$$\int_{-\infty}^{\infty}\delta^{-2r}\left(\frac{x_2 - iy_2'}{x_2 + iy_2'}\right)^{k/2} e(-y_2x_2)\,dx_2 = \frac{y_2^{r-1}(-1)^{k/2}}{\Gamma\left(r+\dfrac{k}{2}\right)}\left(\frac{\pi}{y_2'}\right)^r W_{k/2,\,1/2-r}(4\pi y_2y_2') ,$$

in terms of Whittaker's solution to the confluent hypergeometric equation. Therefore, we have proved that

$$W^{\pm}(y_1, y_2; s, r) = (y_1y_2)^{4-s}\times$$

$$\int_{\mathbf{R}^3}\frac{\sqrt{-\det(X+iE)}}{|\det(X+iE)|^s}e(\pm y_1x_1)e(y_2x_2')\,W_{k/2,\,1/2-r}(4\pi y_2y_2')\mathbf{v}\sigma(\kappa)\,dX . \quad (3.15)$$

It may be checked that this integral is absolutely convergent to the region $\mathrm{re}(s + r) > 5/2$ and $\mathrm{re}(s - r) > 3/2$. This depends on the asymptotics of the confluent hypergeometric function as $y \to 0$, which are set forth in the remark below. The details of the estimation are similar to Proposition 3.1 and will be omitted.

The functional equation (3.13) follows from the well-known property

$$W_{k,r}(y) = W_{k,-r}(y) \quad (3.16)$$

of the confluent hypergeometric function. Now, by Gradshteyn and Ryzhik [8] (9.237.2) and (3.16), we have

$$W_{k/2,(1-k)/2}(y) = y^{k/2}e^{-y/2} .$$

Thus by (3.1), we obtain (3.14). □

Remark. It may seem paradoxical that the region of convergence for the integral (3.15) predicted by the theorem when $r = k/2$ is $\mathrm{re}(s) > (3 + k)/2$, whereas we have proved in Proposition 3.1 that the integral (3.1), to which (3.15) specializes when $r = k/2$, actually converges for $\mathrm{re}(s) > 2$. The explanation for this is as follows. We have, by Whittaker and Watson [21] that

$$W_{k/2,m}(y) = \frac{\Gamma(-2m)}{\Gamma(\frac{1}{2} - m - k/2)}M_{k/2,m}(y) + \frac{\Gamma(2m)}{\Gamma(\frac{1}{2} + m - k/2)}M_{k/2,-m}(y)$$

whenever m is not an integer, where asymptotically the confluent hypergeometric function

$$M_{k/2,m}(y) \sim y^{m+1/2}$$

as $y \to 0$. The point is that when $m = \frac{1}{2}(1-k)$, the second factor disappears, and the asymptotics of the confluent hypergeometric function as $y \to 0$ are then such that the integral (3.15) converges far to the left of its usual region!

This phenomenon is related to the reducibility of the principal series representation discussed in the remark following Proposition 3.6 below.

Our objective is to obtain the analytic continuation of (3.1) further to the left. Let us decompose the vector space $\mathbf{V}$ into eigenspaces for the characters of the subgroup of all elements of K of the form

$$\begin{pmatrix} 1 & & & \\ & \cos\theta & & \sin\theta \\ & & 1 & \\ & -\sin\theta & & \cos\theta \end{pmatrix}. \tag{3.17}$$

If τ_n is the character which has value $e(-n\theta)$ on the matrix (3.17), then

$$\tau_n(\kappa_0) = \frac{(x_1 + iy_1')^{n/2}}{(x_1 - iy_1')^{n/2}}.$$

Thus we write $\mathbf{V} = \bigoplus_n \mathbf{V}_n$ where for $\mathbf{u} \in \mathbf{V}_n$ we have

$$\mathbf{u}\sigma_z = \frac{(x_1 + iy_1')^{n/2}}{(x_1 - iy_1')^{n/2}}\mathbf{u}.$$

Let $\mathbf{v} = \sum \mathbf{v}_n$ where $\mathbf{v}_n \in \mathbf{V}_n$. Then

$$V^{\pm}(y_1, y_2; s, r; \mathbf{v}, \sigma) = \sum_n V^{\pm}(y_1, y_2; s, r; \mathbf{v}_n, \sigma). \tag{3.18}$$

Also, let

$$\hat{V}^{\pm}(y_1, y_2; s, r; \mathbf{v}_n, \sigma) =$$
$$2^{-s}\pi^{-s}\Gamma(\tfrac{1}{2}(\mp r + s \mp n \mp 1))\Gamma(\tfrac{1}{2}(\pm r + s \mp n))\, V^{\pm}(y_1, y_2; s, r; \mathbf{v}_n, \sigma).$$

Proposition 3.4. *$W^{\pm}(y_1, y_2; s, r)$ has analytic continuation to the region* $\mathrm{re}(r) > 1/2$, $\mathrm{re}(s) > 2$. *We have the individual functional equations*

$$\hat{V}^{\pm}(y_1, y_2; s, r; \mathbf{v}_n, \sigma) = \hat{V}^{\pm}(y_1, y_2; r + 3/2, s - 3/2; \mathbf{v}_n, \sigma). \tag{3.19}$$

Proof. Since the gamma function has no zeros, it is sufficient to prove the analytic continuation of $\hat{V}^{\pm}$.

We have $J = J_1 J_2$ where

$$J_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let us substitute $X \to E(-x_2)X^{\mathrm{T}}E(-x_2)$ in (3.12), and then write $X = X_0 + X_1$ where $X_0 = \begin{pmatrix} x_3 & x_4 \\ x_3 & 0 \end{pmatrix}$ and $X_1 = U_1(x_1)$. Since

$$J_2\begin{pmatrix} E & X_1 \\ & E \end{pmatrix} = \begin{pmatrix} E & X_1 \\ & E \end{pmatrix} J_2,$$

we obtain

$$V^{\pm}(y_1, y_2; s, r; \mathbf{v}_n, \sigma) =$$

$$(-1)^{k/2} y_1^{4-s} y_2^{5-r-s} \int_{\mathbf{R}}\int_{\mathbf{R}^3} \mathscr{I}_{\mathbf{v}_n}\left(J_1\begin{pmatrix} E & X_1 \\ & E \end{pmatrix} J_2 \begin{pmatrix} E & X_0 \\ & E \end{pmatrix}\begin{pmatrix} E(x_2) & \\ & {}^{\mathrm{T}}E(-x_2)\end{pmatrix}\right)$$
$$\times \sqrt{-\det(X + iE(x_2)^{\mathrm{T}}E(x_2))} e(\pm y_1 x_1) e(y_2 x_2)\, dX\, dx_2 \, .$$

Now let X', x_2' and Y' be determined so that

$$J_2 \begin{pmatrix} E & X_0 \\ & E \end{pmatrix}\begin{pmatrix} E(x_2) & \\ & {}^{\mathrm{T}}E(-x_2)\end{pmatrix} =$$
$$\begin{pmatrix} E & X' \\ & E \end{pmatrix}\begin{pmatrix} E(x_2') & \\ & {}^{\mathrm{T}}E(-x_2')\end{pmatrix}\begin{pmatrix} Y' & \\ & {}^{\mathrm{T}}Y'^{-1}\end{pmatrix}\kappa_1 \, , \tag{3.20}$$

where κ_1 is a unitary similitude, $Y' = \sqrt{y_1'}\begin{pmatrix} y_2' & \\ & 1\end{pmatrix}$, and $X' = \begin{pmatrix} x_4' & x_3' \\ x_3' & x_1'\end{pmatrix} = X_0' + X_1'$, with $X_0' = \begin{pmatrix} x_3' & x_4' \\ x_3' & 0\end{pmatrix}$ and $X_1' = U_1(x_1')$. Note that $y_1', y_2', x_1', x_2', x_3'$ and x_4' are independent of x_1. Let us show how they may be calculated. Let R_1, R_2, R_3 and R_4 be the four vectors in $\mathbf{R}^4$ which are the rows of the left hand side of (3.20), and let R_1', R_2', R_3' and R_4' be the rows of the matrix which is the product of the first three factors on the right hand side. Clearly R_3 and R_3' have the same length, and therefore we have

$$y_1'^{-1} y_2'^{-2} = 1 + x_2^2 + x_3^2 + (x_4 - x_2 x_3)^2 \, .$$

Also, comparing the lengths of $R_3 \wedge R_4$ and $R_3' \wedge R_4'$ in $\wedge^2 \mathbf{R}^4 = \mathbf{R}^6$, we obtain

$$y_1'^{-2} y_2'^{-2} = (1 + x_2^2)^2 + x_4^2 \, .$$

Solving these equations, we have

$$y_1' = \frac{1 + x_2^2 + x_3^2 + (x_4 - x_2 x_3)^2}{(1 + x_2^2)^2 + x_4^2} \, , \tag{3.21}$$

$$y_2' = \frac{\sqrt{(1 + x_2^2)^2 + x_4^2}}{1 + x_2^2 + x_3^2 + (x_4 - x_2 x_3)^2} \, . \tag{3.22}$$

We also require a value for x_1', which may be obtained as follows. Let $\langle\, , \rangle$ denote the Euclidean inner product on $\mathbf{R}^4$. We have

$$x_1' = y_1' \langle R_2, R_4 \rangle - (y_1' y_2')^2 \langle R_2, R_3 \rangle \langle R_3, R_4 \rangle \, .$$

The inner products may be evaluated by substituting R_i' for R_i, and we obtain

$$x_1' = \frac{x_4 x_2^2 - x_4 x_3^2 - 2x_2 x_3 - 2x_2^3 x_3}{(1 + x_2^2)^2 + x_4^2} \, . \tag{3.23}$$

Now substituting $x_1 \to x_1 - x_1'$, we obtain

$$V^{\pm}(y_1, y_2; s, r; \mathbf{v}_n, \sigma) = (-1)^{k/2} y_1^{4-s} y_2^{5-r-s} \times$$

$$\int_{\mathbf{R}} \int_{\mathbf{R}^3} \mathscr{I}_{\mathbf{v}_n}\left(J_1 \begin{pmatrix} E & X_1 \\ & E \end{pmatrix} \begin{pmatrix} E & X_0' \\ & E \end{pmatrix} \begin{pmatrix} E(x_2') & \\ & {}^{\mathsf{T}}E(-x_2') \end{pmatrix} \begin{pmatrix} Y' & \\ & {}^{\mathsf{T}}Y'^{-1} \end{pmatrix} \kappa_1 \right)$$

$$\times \sqrt{-\det(X - U_1(x_1') + iE(x_2)^{\mathsf{T}}E(x_2))}\, e(\pm y_1 x_1) e(\mp y_1 x_1') e(y_2 x_2)\, dX\, dx_2 \,.$$

Observe that, because of the independence of (3.11) on x_2 and X, we have

$$\mathscr{I}_{\mathbf{v}}\left(J_1 \begin{pmatrix} E & X_1 \\ & E \end{pmatrix} \begin{pmatrix} E & X_0' \\ & E \end{pmatrix} \begin{pmatrix} E(x_2') & \\ & {}^{\mathsf{T}}E(-x_2') \end{pmatrix} \begin{pmatrix} Y' & \\ & {}^{\mathsf{T}}Y'^{-1} \end{pmatrix} \kappa_1 \right) =$$

$$\mathscr{I}_{\mathbf{v}}\left(J_1 \begin{pmatrix} E & X_1 \\ & E \end{pmatrix} \begin{pmatrix} Y' & \\ & {}^{\mathsf{T}}Y'^{-1} \end{pmatrix} \kappa_1 \right).$$

We have

$$J_1 \begin{pmatrix} E & X_1 \\ & E \end{pmatrix} \begin{pmatrix} Y' & \\ & {}^{\mathsf{T}}Y'^{-1} \end{pmatrix} = \begin{pmatrix} Y'' & \\ & {}^{\mathsf{T}}Y''^{-1} \end{pmatrix} \begin{pmatrix} E & U_1(-x_1) \\ & E \end{pmatrix} \kappa_0 \,,$$

where, denoting $\Delta = \sqrt{x_1^2 + y_1'^2}$,

$$Y'' = \sqrt{\Delta^{-2} y_1'} \begin{pmatrix} \Delta y_2' & \\ & 1 \end{pmatrix},$$

and

$$\kappa_0 = \begin{pmatrix} 1 & & & \\ & x_1 \Delta^{-1} & & -y_1' \Delta^{-1} \\ & & 1 & \\ & y_1' \Delta^{-1} & & x_1 \Delta^{-1} \end{pmatrix}.$$

Thus

$$V^{\pm}(y_1, y_2; s, r; \mathbf{v}_n, \sigma) = (-1)^{k/2} y_1^{4-s} y_2^{5-r-s} \int_{\mathbf{R}} \int_{\mathbf{R}^2} (y_1' y_2')^s y_2'^r \times$$

$$\left\{ \int_0^\infty (x_1^2 + y_1'^2)^{(r-s)/2} \sqrt{-\det(X - U_1(x_1') + iE(x_2)^{\mathsf{T}}E(x_2))}\, e(\pm y_1 x_1) \mathbf{v}_n \sigma(\kappa_0)\, dx_1 \right\}$$

$$\times \sigma(\kappa_1) e(\mp y_1 x_1') e(y_2 x_2)\, dX_0\, dx_2 \,. \quad (3.24)$$

It follows from (3.23) that

$$\det(X - U_1(x_1') + iE(x_2)^{\mathsf{T}}E(x_2)) =$$

$$\frac{x_1((1 + x_2^2)^2 + x_4^2) + i(1 + x_2^2 + x_3^2 + (x_4 - x_2 x_3)^2)}{x_4 - i(1 + x_2^2)}$$

$$= (x_4 + i(1 + x_2^2))(x_1 + iy_1') \,.$$

Thus (3.24) equals

$$(-1)^{k/2} y_1^{4-s} y_2^{5-r-s} i^n \int_{\mathbf{R}} \int_{\mathbf{R}^2} (y_1' y_2')^s y_2'^r \sqrt{x_4 + i(1+x_2^2)} \times$$
$$\left\{ \int_0^\infty (y_1' - ix_1)^{(r-s+n+1)/2} (y_1' + ix_1)^{(r-s-n)/2} e(\pm y_1 x_1)\, dx_1 \right\}$$
$$\mathbf{v}_n \sigma(\kappa_1) e(\mp y_1 x_1') e(y_2 x_2)\, dX_0\, dx_2 \,. \quad (3.25)$$

This may be evaluated explicitly by Gradshteyn and Ryzhik [8] (3.384.9). The two cases V^+ and V^- are handled separately. We have

$$\int_{-\infty}^{\infty} (y_1' - ix_1)^{(r-s+n+1)/2} (y_1' + ix_1)^{(r-s-n)/2} e(y_1 x_1)\, dx_1 = -\left(\frac{\pi y_1}{y_1'}\right)^{(-2r+2s-1)/4} \times$$
$$y_1^{-1} \Gamma(\tfrac{1}{2}(-r+s+n))^{-1} W_{(2n+1)/4,(2r-2s+3)/4}(4\pi y_1 y_1') \,, \quad (3.26)$$

and

$$\int_{-\infty}^{\infty} (y_1' - ix_1)^{(r-s+n+1)/2} (y_1' + ix_1)^{(r-s-n)/2} e(-y_1 x_1)\, dx_1 = \left(\frac{\pi y_1}{y_1'}\right)^{(-2r+2s-1)/4} \times$$
$$y_1^{-1} \Gamma(\tfrac{1}{2}(-r+s-n-1))^{-1} W_{(-2n-1)/4,(2r-2s+3)/4}(4\pi y_1 y_1') \,. \quad (3.27)$$

Therefore

$$\hat{V}^+(y_1, y_2; r, s; \mathbf{v}_n, \sigma) = -(-1)^{k/2} \pi^{(-2r-2s-1)/4} \Gamma(\tfrac{1}{2}(r+s+n+1)) \times$$
$$y_1^{-r/2-s/2+11/4} y_2^{5-r-s} i^n \int_{\mathbf{R}} \int_{\mathbf{R}^2} y_1'^{(2r+2s+1)/4} y_2'^{r+s} \sqrt{x_4 + i(1+x_2^2)} \times$$
$$W_{(-2n-1)/4,(2r-2s+3)/4}(4\pi y_1 y_1') \mathbf{v}_n \sigma(\kappa_1) e(y_1 x_1') e(y_2 x_2)\, dX_0\, dx_2$$

and

$$\hat{V}^-(y_1, y_2; r, s; \mathbf{v}_n, \sigma) = (-1)^{k/2} \pi^{(-2r-2s-1)/4} \Gamma(\tfrac{1}{2}(r+s-n)) \times$$
$$y_1^{-r/2-s/2+11/4} y_2^{5-r-s} i^n \int_{\mathbf{R}} \int_{\mathbf{R}^2} y_1'^{(2r+2s+1)/4} y_2'^{r+s} \sqrt{x_4 + i(1+x_2^2)} \times$$
$$W_{(2n+1)/4,(2r-2s+3)/4}(4\pi y_1 y_1') \mathbf{v}_n \sigma(\kappa_1) e(-y_1 x_1') e(y_2 x_2)\, dX_0\, dx_2 \,.$$

These integrals may be shown to be absolutely convergent if $\mathrm{re}(r) > 1/2$, $\mathrm{re}(s) > 2$, and thus give the analytic continuation of $V^\pm(y_1, y_2; r, s; \mathbf{v}_n, \sigma)$ to this region. The functional equation (3.19) now follows from (3.16). □

Proposition 3.5. *The Whittaker function $W^\pm(y_1, y_2; s; \mathbf{v}, \sigma)$ has analytic continuation to all $\mathrm{re}(s) > 3/2$.*

Proof. The proof consists of pointing out that among the preceding integral expressions for the Whittaker functions, we may assemble one which is valid for $3/2 < \mathrm{re}(s) < 5/2$. (Of course for larger s, the original integral (3.1) is convergent.) First assume that $2 < \mathrm{re}(s) < 5/2$. We may use (3.14) and (3.19) to write $W^\pm(y_1, y_2; s)$ as a linear combination of functions $V^\pm(y_1, y_2; \tfrac{1}{2}(k+3), s - \tfrac{3}{2}; \mathbf{v}_n, \sigma)$. The coefficients in this sum will be ratios of Gamma functions. Note that for s in the range $3/2 < \mathrm{re}(s) < 5/2$, the Gamma functions in the numerators may have poles if $s = 2$. However, whenever this occurs, there will be poles among in the Gamma functions in the denominators to balance out, so the coefficients in this sum are holomorphic for the range in question.

Now we note that by Proposition 3.3 (i) the integral (3.12) for $V^{\pm}(y_1, y_2; s, r; \mathbf{v}_n, \sigma)$ is convergent when $3/2 < \mathrm{re}(s) < 5/2$. Thus we obtain an expression for $W^{\pm}(y_1, y_2; s)$ which is valid in this region. □

Proposition 3.6. *If* $\mathrm{re}(s) > 3/2$, *then there exists a Schwartz function* $\xi(y_1, y_2)$ *on* $\mathbf{R}^2$, *and a constant* C *such that*

$$W(y_1, y_2; s; \mathbf{v}, \sigma) < (y_1 y_2)^{-C} \xi(y_1, y_2) .$$

Proof. The prototype for this result is Lemma 8.3.3 of Jacquet, Piatetski-Shapiro and Shalika [11], which asserts a similar bound for Whittaker functions associated with *unitary* generic representations of $GL(n, \mathbf{R})$. The argument extends without difficulty to the metaplectic group. However, it must be explained why it is not necessary to assume that the representations of the metaplectic group with which our Whittaker functions are associated are unitary.

We will explain below that the generic representation associated with the Whittaker function $W(y_1, y_2; s; \mathbf{v}, \sigma)$ is unitary when $\mathrm{re}(s) = 2$. However, it is actually not necessary to have a unitary representation, and the Proposition is true even if $\mathrm{re}(s) \neq 2$. Let us explain why this is the case. The assumption that π is unitary is only used in [11] in their Lemma 8.3.1 to show that the Whittaker function is bounded by a polynomial function of y_1 and y_2. In our case, this is known because of the absolutely convergent integral expressions (3.1) if $\mathrm{re}(s) > 2$, or that described in the proof of Proposition 3.5 if $3/2 < \mathrm{re}(s) < 5/2$. Once this is known, the key point in the Proof of Lemma 8.3.3 of [11] is the formula (8.3.4) which expresses the Whittaker function as an integral of itself times the Fourier transform of a compactly supported smooth function. This of course works in our context also. □

Remark. It may not be superfluous to remark that when $\mathrm{re}(s) = 2$, our Whittaker functions actually *are* associated with unitary representations of the metaplectic group. Let us pause to explain why this is true.

Let $G = GSp(4, \mathbf{R})$, and let $\tilde{G}$ be the metaplectic group, which is the double cover of G. It is well-known that the inclusion map from the Borel subgroup B of elements of the form

$$b = \begin{pmatrix} E(x_2) & \\ & {}^{\mathsf{T}}E(-x_2) \end{pmatrix} \begin{pmatrix} Y & X{}^{\mathsf{T}}Y^{-1} \\ & {}^{\mathsf{T}}Y^{-1} \end{pmatrix}$$

in G lifts to an inclusion $i: B \to \tilde{G}$. If r and s are given, we define a representation $\pi_{r,s}$ of $\tilde{G}$ acting by right translation on the space of complex-valued functions $\mathscr{I}$ satisfying the following generalization of (3.11):

$$\mathscr{I}\left(\begin{pmatrix} E(x_2) & \\ & {}^{\mathsf{T}}E(-x_2) \end{pmatrix} \begin{pmatrix} Y & X{}^{\mathsf{T}}Y^{-1} \\ & {}^{\mathsf{T}}Y^{-1} \end{pmatrix} \tilde{g}\right) = |\det Y|^s y_2^r \mathscr{I}(\tilde{g}) .$$

This is a principal series representation of $\tilde{G}$. A Whittaker functional is defined on this space by an obvious generalization of the integral (3.12) if $\mathrm{re}(s) > 2$, $\mathrm{re}(s - r) > 5/2$, and extends to all r and s by analytic continuation. Each component of the vector-valued function $\mathscr{I}_{\mathbf{v}}$ is an element of the space of $\pi_{r,s}$. Letting $s = 2$ and $r = k/2$ and applying this functional to these components as above gives the Whittaker function in question, according to (3.14).

When $r = k/2$, the representation $\pi_{s,r}$ is reducible. This may be seen as follows. The Borel subgroup B is contained in the standard maximal parabolic subgroup P of G, which has a decomposition $P = MAN$ with N unipotent, $M \sim SL(2, \mathbf{R})$, and

$$A = \left\{ \begin{pmatrix} y_1 & & & \\ & y_1 & & \\ & & y_0 & \\ & & & y_0 \end{pmatrix} \middle| y_1, y_2 > 0 \right\}.$$

We regard the representation $\pi_{s,r}$ as obtained by parabolic induction in stages, and consider the intermediate representation of P. This representation $\pi_{s,r}^P$ of P obtained by induction will be a principal series representation of M twisted by a character $\chi_s \delta^{1/2}$ of A, where δ is the ratio between right and left Haar measures on the group P. If $r = k/2$, the representation of M will be reducible, and its generic composition factor is a subrepresentation which is the holomorphic discrete series representation associated with the weight k. Consequently, the induced representation $\pi_{r,s}$ of $\tilde{G}$ is reducible when $r = k/2$, and has a unique generic irreducible composition factor. The Whittaker function $W(y_1, y_2; s)$ arises from a Whittaker functional on this generic representation.

The reducibility of the representation $\pi_{s,r}$ when $r = k/2$ is related to the phenomenon observed in the remark following the proof of Proposition 3.3.

If $\mathrm{re}(s) = 2$, then the character χ_s will be unitary, and so the generic composition factor representation $\pi_{s,r}^P$ is unitary. Consequently, the generic composition factor of $\pi_{s,r}$ is unitary.

Let us recall some basic facts about matrix coefficients on a compact Lie group, which in our case will be the maximal compact subgroup $K = U(2)$. By definition, a matrix coefficient is a function of the form

$$\phi(\kappa) = T(\mathbf{v}\sigma(\kappa)), \tag{3.28}$$

where $\sigma: K \to GL(\mathbf{V})$ is a finite-dimensional representation, $\mathbf{v} \in \mathbf{V}$ and T is a linear functional on $\mathbf{V}$. Matrix coefficients form a ring, since the class of finite-dimensional representations is closed under direct sum and tensor product. We will denote this ring by $\mathscr{R}$. Also, let $\mathscr{R}_0$ and $\mathscr{R}_k$ be the additive subgroups consisting of functions $\phi \in \mathscr{R}$ which satisfy, respectively, $\phi(\kappa_0 \kappa) = \phi(\kappa)$ and $\phi(\kappa_0 \kappa) = \rho_k(\kappa_0)\phi(\kappa)$ for $\kappa_0 \in SO(2)$. Then $\mathscr{R}_0$ is a subring of $\mathscr{R}$, and $\mathscr{R}_k$ is an $\mathscr{R}_0$-module. Specifically, $\mathscr{R}_k$ is the set of all functions of the form $\kappa \to T(\mathbf{v}\sigma(\kappa))$ where the vector $\mathbf{v}$ satisfies (1.12).

As usual, we will say that one element ϕ of the ring $\mathscr{R}$ *divides* another ϕ' if there is an element ϕ'' such that $\phi' = \phi\phi''$.

Proposition 3.7. *Any continuous function ϕ_0 on K may be uniformly approximated arbitrarily well by an element of $\mathscr{R}$. Moreover, if ϕ_0 satisfies $\phi_0(\kappa_0\kappa) = \rho_k(\kappa_0)\phi_0(\kappa)$, then it may be approximated by an element of $\mathscr{R}_k$.*

Proof. This is an immediate consequence of the Peter-Weyl Theorem. □

Proposition 3.8. *Let κ be as in* (3.3) *and A, B as in* (3.7). *Then the functions*

$$\phi_1 = \frac{1}{\sqrt{(\alpha^2+1)(\beta^2+1)}}, \qquad \phi_2 = \frac{(\alpha^2\beta^2 - 1)(c^2\beta + d^2\alpha)}{(\alpha^2+1)^2(\beta^2+1)^2},$$

$$\phi_3 = \frac{cd(\beta - \alpha)(1 - \alpha\beta)}{(\alpha^2+1)(\beta^2+1)} \tag{3.29}$$

of κ are in $\mathscr{R}_0$.

Proof. Every entry in the matrices A or B is an element of the ring $\mathscr{R}$, and so $\det(B)$ is an element of $\mathscr{R}$. It is clearly independent of U in (3.7), and so this function actually is in $\mathscr{R}_0$. This is the first function ϕ_1. Also, every matrix entry in $A^{\mathrm{T}}B$ is a matrix coefficient of K and is obviously independent of U, and so these are elements of $\mathscr{R}_0$. These four functions include

$$\frac{c^2\alpha}{\alpha^2+1}+\frac{d^2\beta}{\beta^2+1}, \qquad \frac{d^2\alpha}{\alpha^2+1}+\frac{c^2\beta}{\beta^2+1}.$$

Finally,

$$\frac{\alpha\beta}{(\alpha^2+1)(\beta^2+1)}=\det(AB)$$

is in $\mathscr{R}_0$, and so, therefore, is

$$\det(AB)\left[\frac{c^2\alpha}{\alpha^2+1}+\frac{d^2\beta}{\beta^2+1}\right]-\det(B)^2\left[\frac{d^2\alpha}{\alpha^2+1}+\frac{c^2\beta}{\beta^2+1}\right].$$

This equals the second function ϕ_2. The third function ϕ_3 is minus the upper right hand entry in $A^{\mathrm{T}}B$. □

Proposition 3.9. *Let $\phi\in\mathscr{R}$, and for $\kappa_0\in K$ let Φ_{κ_0} be the function of real symmetric matrices defined by $\Phi_{\kappa_0}(X)=\phi(\kappa\kappa_0)$, where κ is the element of K such that (in the notation* (3.7)) *$X=-AB^{-1}$, the upper left hand entry of B is zero, and the upper right entry is positive. Then Φ_{κ_0} is an analytic function of X, and can be extended to the region $x_3, x_4\in\mathbf{R}$, $\mathrm{im}(x_1)\leqq\varepsilon$, where ε is any positive constant less than one, and Φ_{κ_0} is bounded (uniformly in κ_0) in this region.*

Proof. It is well-known that every finite-dimensional representation of a compact Lie group extends to the complexified Lie group, and that the matrix coefficients are analytic. In this case, the complexified Lie group of $U(2)$ is $GL(2,\mathbf{C})$. Specifically, we have associated with a matrix $\kappa^{-1}\in K$ as in (3.3) the unitary matrix $A+Bi$. The complexification of K may then be identified with $GL(2,\mathbf{C})$ because if C is a nonsingular invertible matrix, there exists a unique pair of complex matrices A and B such that ${}^{\mathrm{T}}AB={}^{\mathrm{T}}BA$, ${}^{\mathrm{T}}AA+{}^{\mathrm{T}}BB=E$, and $C^{-1}=A+Bi$. Specifically, we may construct A and B by the formulas $A=\frac{1}{2}(C^{-1}+{}^{\mathrm{T}}C)$, $B=\frac{1}{2i}(C^{-1}-{}^{\mathrm{T}}C)$. The matrix C will be unitary if and only if A and B are real. We have defined a mapping $X\to\kappa=\kappa(X)$ from real symmetric matrices to $U(2)$ by requiring that $X=-AB^{-1}$, that the upper left hand entry of B is zero, and that the upper right hand entry of B is positive.

Lemma. *Let Ω be a simply connected neighborhood of the set of real symmetric matrices in the set of complex symmetric matrices such that $\det(X^2+E)\neq 0$ for $X\in\Omega$, and such that $1+x_1^2+x_3^2\neq 0$ for $X=\begin{pmatrix}x_4 & x_3\\ x_3 & x_1\end{pmatrix}\in\Omega$. Then $X\to\kappa(X)$ may be extended to a holomorphic mapping $\Omega\to GL(2,\mathbf{C})$ such that if $X\in\Omega$, then $\kappa(X)^{-1}=A+Bi$, then $X=-AB^{-1}$, and the upper left entry of B is zero.*

Proof. It is easy to check that if $\det(X^2+E)\neq 0$ and $1+x_1^2+x_3^2\neq 0$, then there are exactly four matrices B such that the upper left entry of B is zero, and ${}^{\mathrm{T}}BB=(X^2+E)^{-1}$. We then define $A=-XB$, and $\kappa(X)=(A+Bi)^{-1}$. The problem may therefore be solved locally. It may be solved globally by the principle of monodromy, because Ω is assumed to be simply connected. □

It may be checked that a neighborhood Ω of the region described in the Proposition satisfies the hypotheses of the Lemma. Therefore the definition $\Phi_{\kappa_0}(X) = \phi(\kappa(X))$ gives the required extension. This function is bounded (uniformly in κ_0) because the closure of $\kappa(\Omega)$ is a compact subset of $GL(2, \mathbf{C})$, and because κ_0 is restricted to the compact space K. □

If $z \in \mathbf{R}$, let $\Delta_z = \sqrt{1 + z^2}$, and

$$\kappa_z = \begin{pmatrix} 1 & & & \\ & \Delta_z^{-1} & & -\Delta_z^{-1} z \\ & & 1 & \\ & \Delta_z^{-1} z & & \Delta_z^{-1} \end{pmatrix}. \tag{3.30}$$

Throughout the following discussion, $\phi(\kappa) = T(\mathbf{v}\sigma(\kappa))$ will denote a matrix coefficient of K. If X and κ are related as in (3.2), we will denote $\Phi(X) = \phi(\kappa)$, and in particular, for the three specific matrix coefficients defined by (3.29), we will denote $\Phi_j(X) = \phi_j(\kappa)$, $(j = 1, 2, 3)$. Now for fixed z, if $x_1 \neq -1/z$ and

$$X = \begin{pmatrix} x_4 & x_3 \\ x_3 & x_1 \end{pmatrix},$$

let us denote

$$X(z) = \begin{pmatrix} x_4(z) & x_3(z) \\ x_3(z) & x_1(z) \end{pmatrix}$$

where

$$x_1(z) = \frac{x_1 - z}{1 + zx_1}, \qquad x_3(z) = \frac{\Delta_z x_3}{1 + zx_1}, \qquad x_4(z) = x_4 - \frac{zx_3^2}{1 + zx_1}.$$

It may be checked that

$$\phi(\kappa\kappa_z) = \begin{cases} \Phi(X(z)) & \text{if } 1 + x_1 z \neq 0\,; \\ 0 & \text{otherwise}\,. \end{cases} \tag{3.31}$$

Also, note that

$$\Phi_1(X) = |\det(X + iE)|^{-1}, \tag{3.32}$$

$$\Phi_2(X) = \Phi_2'(X) x_1, \qquad \Phi_2'(X) = \frac{(\alpha^2\beta^2 - 1)}{(\alpha^2 + 1)^2(\beta^2 + 1)^2} = \frac{\det(X)^2 - 1}{|\det(X + iE)|^4}. \tag{3.33}$$

The function Φ_2' is clearly bounded. We have

$$\Phi_3(X) = \frac{x_3(1 - \det(X))}{|\det(X + iE)|^2}. \tag{3.34}$$

It follows that

$$\Phi_1(X(z)) = \frac{1 + x_1 z}{\sqrt{1 + z^2}} \Phi_1(X), \tag{3.35}$$

$$\Phi_2(X(z)) = \Phi_2'(X(z)) \frac{x_1 - z}{1 + zx_1}. \tag{3.36}$$

and

$$\Phi_3(X(z)) = \Delta_z^{-1} x_3(1 + zx_1 + zx_4 - \det(X))\Phi_1(X)^2. \tag{3.37}$$

Suppose that $\mathbf{v}$ satisfies (1.12), so that $\phi(\kappa) = T(\mathbf{v}\sigma(\kappa))$ is in r_k. Then

$$T\{W^{\pm}(y_1, y_2; s)\sigma(\kappa_z)\} = (y_1 y_2)^{4-s} y_2^{k/2} \times$$
$$\int_{\mathbf{R}^3} \frac{\sqrt{-\det(X+iE)}}{|\det(X+iE)|^s} e(\pm y_1 x_1) e(y_2(x_2' + iy_2'))(y_2')^{k/2} \phi(\kappa\kappa_z)\, dX \,. \quad (3.38)$$

provided this integral is absolutely convergent. By Proposition 3.2, this will be the case if $\mathrm{re}(s) > 2$. However, it is useful to know that if the matrix coefficient ϕ is divisible by ϕ_1, then we have absolute convergence farther left. Also, it is useful to have some more precise information on the "polynomial part" of the estimate in Proposition 3.6.

Proposition 3.10. *If $\phi \in \mathscr{R}_k$ is divisible by ϕ_1, then the integral* (3.31) *is absolutely convergent for* $\mathrm{re}(s) > 3/2$. *Assuming* $\mathrm{re}(s) > 3/2$, *if C is any positive constant, there exists a constant C' depending on C such that if $y_2 > C$ and y_1 is arbitrary, then*

$$T\{W^{\pm}(y_1, y_2; s)\sigma(\kappa_z)\} < C' y_1^{4-s} \,.$$

Proof. Estimate

$$e(y_2(x_2' + iy_2'))(y_2')^{k/2} \ll y_2' \,,$$
$$\phi(\kappa\kappa_z) \ll \Phi_1(X(z)) \ll (1 + x_1^2)^{1/2} |\det(X+iE)|^{-1} \,,$$

and apply Proposition 3.1 with $\alpha = \frac{s}{2} - \frac{1}{4}$, $\beta = 1$, $\gamma = -\frac{1}{2}$.

To prove the second part, observe that as long as y_2 is bounded away from zero, these estimates are uniform in y_1 and y_2. Thus if $y_2 > C$, there exists an absolute constant C'' such that

$$T\{W^{\pm}(y_1, y_2; s)\sigma(\kappa_z)\} < C'' y_1^{4-s} y_2^{4-s+k/2} \,.$$

However by Proposition 3.6, as $y_2 \to \infty$, this decays very rapidly, and so we may ignore the factor $y_2^{4-s+k/2}$ in this estimate. □

We will be concerned with the following *Novodvorsky transforms* of the non-degenerate Whittaker functions, which play a role in this theory similar to that of the Mellin transform. We define

$$\mathscr{F}^{\pm}(u, s, y_2) = \mathscr{F}^{\pm}(u, s, y_2; \mathbf{v}, \sigma) =$$
$$\int_0^\infty \int_{-\infty}^\infty \{W^{\pm}((1+z^2)^{-1} y_1, \sqrt{1+z^2}\, y_2; s; \mathbf{v}, \sigma)\sigma(\kappa_z)\} e\left(\frac{\mp y_1 z}{1+z^2}\right) y_1^{u-3/2} \sqrt{1+iz}\, dz \frac{dy_1}{y_1}, \quad (3.37)$$

provided the integral is absolutely convergent. Consequently by (3.31)

$$T\mathscr{F}^{\pm}(u, s, y_2) = \int_0^\infty \int_{-\infty}^\infty \int_{\mathbf{R}^3} (y_1 y_2)^{4-s} y_2^{k/2} (1+z^2)^{2-s/2+k/2} \frac{\sqrt{-\det(X+iE)}}{|\det(X+iE)|^s}$$
$$e(\sqrt{1+z^2}\, y_2(x_2' + iy_2')) e\left(\frac{\pm y_1(x_1 - z)}{1+z^2}\right)(y_2')^{k/2} \Phi(X(z)) y_1^{u-3/2} \sqrt{1+iz}\, dX\, dz \frac{dy_1}{y_1}, \quad (3.38)$$

again provided the integral is convergent.

Proposition 3.11. *The integral* (3.37) *is absolutely convergent for sufficiently large u, and if it is convergent for some particular u, then it is convergent for all larger u. If the matrix coefficient* ϕ *associated with* $\mathbf{v}$ *and* T *is divisible by* ϕ_1, *then the integral* (3.38) *is convergent when* $\mathrm{re}(u - s + 5/2) > 0$.

Proof. The first assertion follows from Proposition 3.6. The second assertion follows from Proposition 3.10. □

Remark. It is important to understand what is being asserted here. We are *not* claiming that the integral (3.38) is absolutely convergent. In fact, the inner integral with respect to X is absolutely convergent, and after this integral is carried out, the outer integrations with respect to z and y_1 become absolutely convergent. However, the integrand is not a positive function, and it is *not* legitimate to interchange the inner integral with respect to X with the outer integrals.

We will also be concerned with the integral

$$\tau(s, y_2; \mathbf{v}, \sigma) = \int_{-\infty}^{\infty} \Delta_z^{-s+k/2} e^{-2\pi y_2 \Delta_z} \sqrt{1 + iz}\,\mathbf{v}\sigma(\eta w^{-1}\kappa_z wJ)\,dz\,. \tag{3.39}$$

It is clear that this integral is convergent for all s and y_2.

Proposition 3.12. *There exists a representation* $\sigma: K \to GL(\mathbf{V})$, *a vector* $\mathbf{v} \in \mathbf{V}$ *satisfying* (1.12), *and a linear functional* T *on* $\mathbf{V}$ *such that both* $T\mathscr{F}^{\pm}(u, s, y_2)$ *have analytic continuation to all u and s such that* $\mathrm{re}(s) > 3/2$, $\mathrm{re}(u - s + 5/2) > 0$. *Moreover, given s and u in this region,* $\mathbf{V}$, $\mathbf{v}$, T *and* y_2 *may be chosen so that* $T\mathscr{F}^{+}(u, s, y_2) \neq 0$. *It may furthermore be arranged so that the function* $T\tau(s, y_2; \mathbf{v}, \sigma)$ *vanishes identically for all s and* y_2.

Proof. We have already shown in Proposition 3.11 that if ϕ_1 divides the matrix coefficient ϕ associated with $\mathbf{v}$ and T, then the integrals (3.38) are convergent for s and u in the indicated region. Let us point out that if ϕ_2 divides ϕ, then $T\tau(s, y_2; \mathbf{v}, \sigma) = 0$ for all s and y_2. Since

$$T\tau(s, y_2; \mathbf{v}, \sigma) = \int_{-\infty}^{\infty} \Delta_z^{-s+k/2} e^{-2\pi y_2 \Delta_z} \sqrt{1 + iz}\,\phi(\eta w^{-1}\kappa_z wJ)\,dz \tag{3.40}$$

it is sufficient to show that $\phi_2(\eta w^{-1}\kappa_z wJ) = 0$ for all z. Let $\kappa = w\eta w^{-1}\kappa_z wJ$. Then

$$\kappa = \begin{pmatrix} & & & -1 \\ -\Delta_z^{-1} z & & \Delta_z^{-1} & \\ & 1 & & \\ \Delta_z^{-1} & & -\Delta_z & \end{pmatrix},$$

so κ^{-1} has the form (3.3) with

$$A = \begin{pmatrix} 0 & -\Delta_z^{-1} z \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -\Delta_z^{-1} \\ -1 & 0 \end{pmatrix}.$$

Now

$$\phi(\eta w^{-1}\kappa_z wJ) = \phi(\kappa) = \Phi(-AB^{-1}) = \Phi\left(\begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix}\right). \tag{3.41}$$

Since $\phi_2 | \phi$, this vanishes by (3.33).

Thus all we have to do is to show that there exists a matrix coefficient ϕ divisible by ϕ_1 and ϕ_2 such that $T\mathscr{F}^+(u, s, y_2) \neq 0$. The difficulty in proving this comes from the lack of absolute convergence which was pointed out in the remark following Proposition 3.11. We will argue by contradiction. Assume, therefore that $T\mathscr{F}^+(u, s, y_2) = 0$ for all matrix coefficients divisible by $\phi_1\phi_2$.

Let v be a fixed, very large real number.

It follows from Proposition 3.6 that the following integral is absolutely convergent:

$$\int_0^\infty T\mathscr{F}^+(u, s, y_2; \mathbf{v}, \sigma) y_2^v \frac{dy_2}{y_2} =$$

$$\int_0^\infty \int_0^\infty \int_{-\infty}^\infty T\{W^+((1+z^2)^{-1} y_1, \sqrt{1+z^2}\, y_2; s; \mathbf{v}, \sigma)\sigma(\kappa_z)\}$$

$$\times\, e\left(\frac{-y_1 z}{1+z^2}\right) y_1^{u-3/2} \sqrt{1+iz}\, y_2^v\, dz \frac{dy_1}{y_1}\frac{dy_2}{y_2}, \quad (3.42)$$

and with our hypotheses, this represents 0. Of course when we assert that the integral is absolutely convergent, we do *not* wish to imply that it has been established that if the integral (3.38) is substituted for $T\mathscr{F}^+$ that the resulting integral is absolutely convergent, and that it is legitimate to interchange the order of integration of X with the other variables. However, it is legitimate to integrate with respect to y_2 before integrating z and y_1, and we will now show that under certain circumstances, it is then legitimate to interchange the X and y_2 integrals. Consider

$$\int_0^\infty T\{W^+((1+z^2)^{-1} y_1, \sqrt{1+z^2}\, y_2; s; \mathbf{v}, \sigma)\sigma(\kappa_z)\} y_2^v \frac{dy_2}{y_2} =$$

$$\int_0^\infty \int_{\mathbf{R}^3} (y_1 y_2)^{4-s} y_2^{k/2} (1+z^2)^{2-s/2+k/4} \frac{\sqrt{-\det(X+iE)}}{|\det(X+iE)|^s}$$

$$e(\sqrt{1+z^2}\, y_2(x_2' + iy_2')) e\left(\frac{y_1 x_1}{1+z^2}\right) (y_2')^{k/2} \Phi(X(z)) y_2^v\, dX \frac{dy_2}{y_2} =$$

$$\Gamma\left(4 - s + \frac{k}{2} + v\right) \int_{\mathbf{R}^3} y_1^{4-s} (1+z^2)^{-v/2} \frac{\sqrt{-\det(X+iE)}}{|\det(X+iE)|^s}$$

$$(x_2' + iy_2')^{-4+s-k/2-v} e\left(\frac{y_1 x_1}{1+z^2}\right) (y_2')^{k/2} \Phi(X(z))\, dX \quad (3.43)$$

provided that $4 - s + \dfrac{k}{2} + v > 0$, and provided that the last integral is absolutely convergent. We will show that this is the case if ϕ is divisible by $\phi_1^{n_1}\phi_3^{n_3}$ where $n_1 > \frac{3}{2} - s + k$ and $n_3 \geqq 4 = s + v + k/2$. Indeed, it is readily established that for fixed z, the function $\Phi_3(X(z))/(x_2' + iy_2')$ is a bounded function of X. It follows that if (v being fixed) ϕ is divisible by $\phi_3^{n_3}$, with n_3 as above, we may estimate the integral by Proposition 3.1 with $\alpha = \dfrac{s}{2} - \dfrac{1}{4} - \dfrac{k}{4} + \dfrac{n_1}{2}$, $\beta = k/2$ and $\gamma = -n_1/2$. Since $k \geqq 2$, we have convergence provided that $n_1 > \frac{3}{2} - s + k$.

When (3.43) is valid, we may substitute this into (3.42) to obtain

$$\int_0^\infty T\mathscr{F}^+(u, s, y_2; \mathbf{v}, \sigma) y_2^v \frac{dy_2}{y_2} =$$

$$\Gamma\left(4 - s + \frac{k}{2} + v\right) \int_0^\infty \int_{-\infty}^\infty \int_{\mathbf{R}^3} (1 + z^2)^{-v/2} \frac{\sqrt{-\det(X + iE)}}{|\det(X + iE)|^s} \times$$

$$(x_2' + iy_2')^{-4+s-k/2-v} e\left(\frac{y_1(x_1 - z)}{1 + z^2}\right)(y_2')^{k/2} \Phi(X(z)) y_1^{u-s+5/2} \sqrt{1 + iz}\, dX\, dz \frac{dy_1}{y_1}.$$

We have proved, with our assumption, that this converges and represents 0 whenever ϕ is sufficiently divisible by ϕ_1 and ϕ_3. (Also, since we want $T\tau(s, y_2; \mathbf{v}, \sigma) = 0$, we are assuming that ϕ is divisible once by ϕ_2.) We have also proved that after the X integral is carried out, the integral with respect to y_1 and z is absolutely convergent, and it is legitimate to interchange the integration and do the y_1 integral before the z integral. Now, by Proposition 3.9 it is legitimate to shift the line of integration with respect to x_1 from the real line upwards a small distance ε. As soon as this is done, due to the exponential decay of the factor $e\left(\frac{y_1 x_1}{1 + z^2}\right)$ and $y_1 \to \infty$, it becomes legitimate to interchange the order of integration with respect to y_1 and X. Another Gamma function $\Gamma(u - s + 5/2)$ appears from the integration with respect to y_1. Then we would like to move the path of integration with respect to x_1 back to the real axis. This is legitimate if ϕ is divisible by $\phi_2^{n_2}$ with $n_2 > u - s + 5/2$, because the vanishing of $\phi_2(X(z))$ when $x_1 = z$ (cf. (3.36)) covers the blowup of the factor $(x_1 - z)^{-5/2+s-u}$ which comes from the y_1 integral. Dropping the two gamma functions and a factor of $(-2\pi i)^{-5/2-u+s}$, we therefore obtain the vanishing for all ϕ sufficiently divisible by ϕ_1, ϕ_2 and ϕ_3 of

$$\int_{-\infty}^\infty \int_{\mathbf{R}^3} (1 + z^2)^{u-s+5/2-v/2} \frac{\sqrt{-\det(X + iE)}}{|\det(X + iE)|^s} (x_2' + iy_2')^{-4+s-k/2-v} \times$$

$$(x_1 - z)^{-5/2+s-u} (y_2')^{k/2} \Phi(X(z)) \sqrt{1 + iz}\, dX\, dz. \tag{3.44}$$

Let us point out that this integral is *absolutely convergent*. To see this, note that we have already assumed that ϕ is divisible by $\phi_2^{n_2}\phi_3^{n_3}$, where n_2 and n_3 are sufficiently large that $(x_1 - z)^{-5/2+s-u}\Phi_2(X(z))^{n_2}$ and $(x_2' + iy_2')^{-4+s-k/2-v}\Phi_3(X(z))^{n_3}$ are bounded. Estimating these factors by 1, the integral splits into two. The integral with respect to z is convergent if v is large, while the integral with respect to X is convergent since ϕ_1 divides ϕ, by Proposition 3.10. Hence integral (3.44) is absolutely convergent.

At this point, we have yet by Proposition 3.7 considerable flexibility in the choice of ϕ. Let us choose a and b to be any constants such that $\Phi_i (i = 1, 2, 3)$ do not vanish on $X = \begin{pmatrix} 0 & b \\ b & a \end{pmatrix}$. We choose ϕ so that Φ approximates a unit mass concentrated at this particular X. Then the integral equals

$$C \int_{-\infty}^\infty (1 + z^2)^{u-s+5/2-v/2} \frac{\sqrt{-\det(X + iE)}}{|\det(X + iE)|^s} (x_2' + iy_2')^{-4+s-k/2-v} \times$$

$$(x_1 - z)^{-5/2+s-u} (y_2')^{k/2} \sqrt{1 + iz} \frac{|1 + zx_1|^3}{(1 + z^2)^3} dz, \tag{3.45}$$

where, for fixed z, X denotes the unique solution of $X(z) = \begin{pmatrix} 0 & b \\ b & a \end{pmatrix}$. The factor $\frac{|1 + zx_1|^3}{(1 + z^2)^3}$ here is the reciprocal of the Jacobian of the transformation (for fixed z) $X \to X(z)$. Now we will obtain a contradiction from our conclusion that this integral vanishes for all v.

We have more precisely

$$x_1 = \frac{a + z}{1 - az}, \qquad x_3 = \frac{b\Delta_z}{1 - az}, \qquad x_4 = \frac{zb^2}{1 - az},$$

$$x_2' = \frac{-b(a + z + zb^2)}{\Delta_z(1 + a^2 + b^2)},$$

$$y_2' = \frac{|1 - az|\sqrt{a^2 + (1 + b^2)^2}}{\Delta_z(1 + a^2 + b^2)},$$

so that

$$|x_2' + iy_2'|\sqrt{1 + z^2} = \sqrt{b^2(a + z(1 + b^2))^2 + (1 - az)^2(a^2 + (1 + b^2)^2)}.$$

This quadratic expression has a minimum, as a function of z at the value $z = a/(a^2 + b^2 + b^4)$. Consequently if v is very large, the integral (3.45) will be approximated by a constant times the value of the integrand at this value of z. It is clear that this does not vanish for all v. This contradiction concludes the proof of Proposition 3.12. □

Proposition 3.13. *Given s such that* $\mathrm{re}(s) > 3/2$, *we may choose* $\mathbf{V}$, $\mathbf{v}$, T *and* y_2 *so that both* $T\mathcal{F}^{\pm}(u, s, y_2)$ *have analytic continuation to the region of Proposition* 3.12, *and so that* $T\tau(s, y_2; \mathbf{v}, \sigma) \neq 0$.

Proof. We take ϕ divisible by ϕ_1. Then Proposition 3.11 guarantees the analytic continuation of $T\mathcal{F}^{\pm}$. The function $\Phi_1\left(\begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix}\right)$ is not identically zero, and by Proposition 3.7 we may choose an element ϕ' of $\mathcal{R}_k$ so that if $\phi = \phi_1\phi'$, then the corresponding matrix function Φ does not vanish identically on matrices of the form $\begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix}$. Then by (3.41), $\phi(\eta w^{-1}\kappa_z w J)$ is not identically zero as a function of z. Thus by (3.40), as a function of y_2, $T\tau(s, y_2; \mathbf{v}, \sigma)$ is the Laplace transform of a function which is not identically zero, and hence by the invertibility of the Laplace transform, $T\tau(s, y_2; \mathbf{v}, \sigma) \neq 0$ for some y_2. □

This completes the theory of the *nondegenerate* Whittaker functions which occur. There are also certain *degenerate* Whittaker functions, which we now define.

We will also need to discuss the following *degenerate Whittaker functions*. Let $W^0(y_1, y_2; s) = W^0(y_1, y_2; s; \mathbf{v}, \sigma) =$

$$(y_1 y_2)^{4-s} y_2^{k/2} \int_{\mathbf{R}^3} \frac{\sqrt{-\det(X + iE)}}{|\det(X + iE)|^s} e(y_2(x_2' + iy_2'))(y_2')^{k/2} \mathbf{v}\sigma(\kappa)\, dX\,. \qquad (3.46)$$

It follows from Proposition 1.1 that this integral is convergent if re(s) > 2.

Proposition 3.14. *The degenerate Whittaker function* W^0 $(y_1, y_2; s)$ *has analytic continuation to* $\mathrm{re}(s) > 3/2$. *For fixed* y_1, *it decays faster than any polynomial as* $y_2 \to \infty$.

Proof. Let us start with s in the range $2 < \mathrm{re}(s) < 5/2$. Imitating the proof of Proposition 3.3, we have

$$W^0(y_1, y_2; s) = \pi^{-r}\Gamma\left(r + \frac{k}{2}\right)V^0\left(y_1, y_2; s, \frac{k}{2}\right)$$

where, by analogy with (3.12),

$$V^0(y_1, y_2; s, r) = (-1)^{k/2} y_1^{4-s} y_2^{5-r-s} \int_{\mathbf{R}}\int_{\mathbf{R}^3} \mathscr{I}\left(J\begin{pmatrix} E(x_2) & \\ & {}^{T}E(-x_2)\end{pmatrix}\begin{pmatrix} E & X \\ & E\end{pmatrix}\right)$$
$$\times \sqrt{-\det(X + iE)}\, e(y_2 x_2)\, dX\, dx_2 \,. \tag{3.47}$$

Proceeding as in the proof of Proposition 3.4, we have the following analog of (3.25):

$$(-1)^{k/2} y_1^{4-s} y_2^{5-r-s} i^n \int_{\mathbf{R}}\int_{\mathbf{R}^2} (y_1' y_2')^s y_2'^r \sqrt{x_4 + i(1 + x_2^2)} \times$$
$$\left\{\int_0^\infty (y_1' - ix_1)^{(r-s+n+1)/2} (y_1' + ix_1)^{(r-s-n)/2}\, dx_1\right\} \mathbf{v}_n \sigma(\kappa_1) e(y_2 x_2)\, dX_0\, dx_2 \,,$$

where y_1' and y_2' are given by (3.21) and (3.22). Now the expression in braces equals $y_1'^{r-s+3/2}$ times a beta integral which can never vanish unless $r + s$ is a half-integer, which of course it will not be if $r = k/2$ and s is in the range $3/2 < \mathrm{re}(s) < 5/2$. Thus we must consider the integral

$$y_1^{4-s} y_2^{5-r-s} \int_{\mathbf{R}}\int_{\mathbf{R}^2} y_1'^{r+3/2} y_2'^{r+s} \sqrt{x_4 + i(1 + x_2^2)}\, \mathbf{v}_n \sigma(\kappa_1) e(y_2 x_2)\, dX_0\, dx_2 \,.$$

We may now continue to imitate the proof of Proposition 3.4, to do the integrals with respect to x_3, x_4 and x_2 in that order. Each step is similar to the one just carried out, for the integration with respect to x_1. At each step, it is necessary to break the vector up into a sum of vectors which transform according to a character of the root group of K associated with the variable (x_3, x_4 or x_2) at hand. This gives us a decomposition analogous to (3.18), and each summand is to be treated separately. The x_3 and x_4 integrals give beta functions times powers of (progressively simpler) quadratic forms in the remaining variables. The final x_2 integral gives a confluent hypergeometric integral. At this point, we have an expression which gives the analytic continuation of the degenerate Whittaker function, and the exponential decay in y_2 comes from the exponential decay of the confluent hypergeometric function.

We omit the details, but compare the proof of (5.23) in our previous paper [3] through a succession of integrals (5.13), (5.14), (5.15) and (5.16) in which the variables x_1, x_3, x_4 and x_2 are successively eliminated, the final integral yielding a K-Bessel function, which is a special case of the confluent hypergeometric function.

Remark. It is also possible to get the decay of the degenerate Whittaker function by imitating the proof of Lemma 8.3.3 of [11]. It is still possible to show that the degenerate Whittaker function may be represented as the convolution of itself with a compactly supported smooth function (essentially since the function $\mathscr{I}$ from which it is constructed may be so represented). The analog of formula (8.3.4) remains valid, but the character of the unipotent subgroup is only nondegenerate in x_2 (not x_1). Thus one obtains an expression for the Whittaker function as an integral of itself times a function of the form $\hat{f}(0, y_2)$, where $\hat{f}$ is the Fourier transform of a smooth compactly supported function. Hence their argument gives rapid decay in y_2, but not y_1.

We will encounter the following transforms of the degenerate Whittaker functions. Let

$$\mathscr{M}(s, y_2; \mathbf{v}, \sigma) = \int_{-\infty}^{\infty} \Delta_z^{2s-8} W^0(1, \Delta_z y_2; s)\sigma(\kappa_z)\sqrt{1+iz}\, dz\,, \tag{3.48}$$

and

$$\hat{\mathscr{M}}(s, y_2; \mathbf{v}, \sigma) = \int_{-\infty}^{\infty} \Delta_z^{2s-8} W^0(1, \Delta_z y_2; s)\sigma(w\kappa_z J)\sqrt{1+iz}\, dz\,. \tag{3.49}$$

It follows from Proposition 3.14 that these integrals converge if re$(s) > 3/2$.

The Whittaker functions actually arise as slight variants of the integral (3.1). We will now prove

$$W^{\pm}(y_1, y_2; s) = W^{\pm}(y_1, y_2; s; \mathbf{v}, \sigma) =$$

$$(y_1 y_2)^s \int_{\mathbf{R}^3} \frac{\sqrt{-\det(X + iY^{\mathsf{T}}Y)}}{|\det(X + iY^{\mathsf{T}}Y)|^s} e(\pm x_1) e(x_2' + iy_2')(y_2')^{k/2} \mathbf{v}\sigma(\kappa)\, dX\,. \tag{3.50}$$

Here the notation is slightly changed. As in (3.1), we are identifying a real symmetric matrix $X = \begin{pmatrix} x_4 & x_3 \\ x_3 & x_1 \end{pmatrix}$ with the point (x_1, x_3, x_4) of $\mathbf{R}^3$. Y, x_2', y_2' and κ are described as follows. As before $Y = \sqrt{y_1}\begin{pmatrix} y_2 & \\ & 1 \end{pmatrix}$, but in place of (3.2) we are writing

$$\begin{pmatrix} & w \\ -w & \end{pmatrix}\begin{pmatrix} Y & X^{\mathsf{T}}Y^{-1} \\ & {}^{\mathsf{T}}Y^{-1} \end{pmatrix} = \begin{pmatrix} Q' & X'^{\mathsf{T}}Q'^{-1} \\ & {}^{\mathsf{T}}Q'^{-1} \end{pmatrix}\kappa \tag{3.51}$$

with $Q' = \sqrt{y_1'}\begin{pmatrix} y_2' & x_2' \\ & 1 \end{pmatrix}$ and κ a unitary similitude (which depends on X and Y).

Indeed, the equivalence between (3.50) and (3.1) follows immediately on substituting $YX^{\mathsf{T}}Y$ for X in (3.50).

If $n_1 \neq 0$ and n_2 is positive, the change of variables in (3.50)

$$(x_1, x_3, x_4) \to (|n_1| x_1, |n_1| n_2 x_3, |n_1| n_2^2 x_4)$$

shows that

$$(y_1 y_2)^s \int_{\mathbf{R}^3} \frac{\sqrt{-\det(X + iY^{\mathsf{T}}Y)}}{|\det(X + iY^{\mathsf{T}}Y)|^s} e(n_1 x_1) e(n_2(x_2' + iy_2'))(y_2')^{k/2} \mathbf{v}\sigma(\kappa)\, dX$$

$$= \begin{cases} (|n_1| n_2)^{s-4} n_2^{-k/2} W^{+}(|n_1| y_1, n_2 y_2; s) & \text{if } n_1 > 0\,; \\ (|n_1| n_2)^{s-4} n_2^{-k/2} W^{-}(|n_1| y_1, n_2 y_2; s) & \text{if } n_1 < 0\,. \end{cases} \tag{3.52}$$

We have similarly

$$W^0(y_1, y_2; s) = (y_1 y_2)^s \int_{\mathbf{R}^3} \frac{\sqrt{-\det(X + iY^\mathrm{T}Y)}}{|\det(X + iY^\mathrm{T}Y)|^s} e(x'_2 + iy'_2)(y'_2)^{k/2} \mathbf{v}\sigma(\kappa)\, dX .$$

The same change of variables as before shows that if n_2 is positive, then

$$(y_1 y_2)^s \int_{\mathbf{R}^3} \frac{\sqrt{-\det(X + iY^\mathrm{T}Y)}}{|\det(X + iY^\mathrm{T}Y)|^s} e(n_2(x'_2 + iy'_2))(y'_2)^{k/2} \mathbf{v}\sigma(\kappa)\, dX$$
$$= (|n_1| n_2)^{s-4} n_2^{-k/2} W^0(n_1| y_1, n_2 y_2; s) \quad (3.53)$$

Note that the left hand side in this equation is independent of n_1. Thus we have the homogeneity property

$$W^0(y_1, y_2; s) = y_1^{4-s} W^0(1, y_2; s) . \quad (3.54)$$

Finally, let us state a result which will not be used in the proof of part (i) of the Theorem, but only in the new proof of part (ii)—Waldspurger's Theorem. We will omit the proof, which may be supplied along the lines of Proposition 3.12.

Proposition 3.15. *There exists a representation $\sigma: K \to GL(\mathbf{V})$, a vector $\mathbf{v} \in \mathbf{V}$ satisfying* (1.12), *and a linear functional T on* $\mathbf{V}$ *such that both $T\mathscr{F}^{\pm}(u, s, y_2)$ have analytic continuation to all u and s such that* $\mathrm{re}(s) > 3/2$, $\mathrm{re}(u - s + 5/2) > 0$, *and such that $\hat{\mathscr{M}}(2, y_2; \mathbf{v}, \sigma) \neq 0$, but $T\tau(s, y_2; \mathbf{v}, \sigma)$ vanishes identically for all s and y_2.* □

4. Möbius inversion for symmetric pairs

Let C and D be the bottom 2×2 blocks of a matrix in $Sp(4, \mathbf{Z})$. Then C and D form a *symmetric pair*: $C^t D$ is symmetric. Moreover C and D are *relatively prime*: if GC and GD are both integral matrices then so is G. Conversely, every relatively prime integral symmetric pair may be realized as the bottom row of an integral symplectic matrix (cf. Maass [14], section 11). The goal of this section is to develop a Möbius inversion formula to pick out these relatively prime pairs from among all the integral symmetric pairs. We also give a formula to do this when the C and D are required to satisfy certain congruence conditions.

If H and C are nonsingular integral matrices, we say that H *divides* C if $H^{-1}C$ is integral. Note that if H divides C then so does HU for any U in $GL(2, \mathbf{Z})$. We write

$$\sum_{H|C} f(H)$$

for the sum over integral matrices H dividing C *modulo* the right multiplication of H by matrices U in $GL(2, \mathbf{Z})$. Of course this notation makes sense only if $f(HU) = f(H)$ for $U \in GL(2, \mathbf{Z})$.

Let μ be the usual Möbius function. Define a function on nonsingular integral matrices, also denoted μ, as follows: If H is in $\Gamma_1 \begin{pmatrix} a & \\ & b \end{pmatrix} \Gamma_1$, where $\Gamma_1 = GL(2, \mathbf{Z})$ and a, b are positive integers, then

$$\mu(H) = \gcd(a, b)\mu(a)\mu(b) .$$

It follows from elementary divisor theory that this is a valid definition, since every integral matrix is contained in such a double coset, and if

$\Gamma_1\begin{pmatrix} a & \\ & b \end{pmatrix}\Gamma_1 = \Gamma_1\begin{pmatrix} a' & \\ & b' \end{pmatrix}\Gamma_1$ then there exist a_1, a_2, b_1, b_2 such that $\gcd(a_1 b_1, a_2 b_2) = 1$ and $a = a_1 a_2$, $b = b_1 b_2$, $a' = a_1 b_2$ and $b' = b_1 a_2$, so that necessarily $\gcd(a, b)\mu(a)\mu(b) = \gcd(a', b')\mu(a')\mu(b')$.

Proposition 4.1. *Let C be a nonsingular integral matrix. Then*

$$\sum_{H|C} \mu(H) = \begin{cases} 1 & \text{if } C \in \Gamma_1; \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

Proof. Neither side of (4.1) is altered if C is multiplied on either the left or the right by an integral unimodular matrix. Therefore we may assume that C is diagonal. Furthermore once C is chosen, we must sum the representatives H modulo the right action of Γ_1. So without loss of generality, we may take $C = \begin{pmatrix} a & \\ & b \end{pmatrix}$, $H = \begin{pmatrix} r & s \\ 0 & t \end{pmatrix}$ with a, b, r and $t > 0$, and s is taken modulo r. Then $H|C$ if and only if $r|a$, $t|b$ and $rt|sb$. Thus

$$\sum_{H|C} \mu(H) = S(a, b)\ ,$$

where

$$S(a, b) = \sum_{\substack{r|a \\ t|b}} \sum_{\substack{s \bmod r \\ rt|sb}} \mu(\gcd(r, s, t))\mu(rt/\gcd(r, s, t))\gcd(r, s, t, rt/\gcd(r, s, t))\ .$$

It is easy to see that if $a = a_1 a_2$, $b = b_1 b_2$ with $\gcd(a_1 b_1, a_2 b_2) = 1$, then

$$S(a, b) = S(a_1, b_1)S(a_2, b_2)\ ,$$

and consequently it is sufficient to prove the Proposition when a and b are both powers of a prime p. Nonzero contributions to the sum arise only when $r = 1$, or when $r = p$ (and $p|a$); computing these directly, the result follows. □

Proposition 4.2. *Let C and D be an integral symmetric pair. Then there exists a nonsingular integral matrix H such that $H|C$ and $H|D$, and such that $H^{-1}C$ and $H^{-1}D$ are relatively prime. H is in Γ_1 if and only if C and D are relatively prime. Furthermore, a nonsingular integral matrix H_1 divides both C and D if and only if $H_1|H$.*

Proof. By elementary divisor theory there exist $U_1 \in GL(2, \mathbf{Z})$ and $U_2 \in GL(4, \mathbf{Z})$ such that

$$(C, D) = U_1(C_1, D_1)U_2\ ,$$

where C_1 is diagonal and $D_1 = 0$. Choosing $H = U_1 C_1$, the first assertion follows. The last two are immediate from this, using the definition of relative primality. □

Given a symmetric pair (C, D), we say that $D \equiv D' \bmod C$ if $D' = D + CS$ for some integral symmetric matrix S. Note that (C, D') is again a symmetric pair; moreover, C and D are relatively prime if and only if C and D' are. We write

$$\sum_{D \bmod C} h(C, D) \qquad \text{and} \qquad \sum_{\substack{D \bmod C \\ \gcd(D, C) = 1}} h(C, D)$$

to denote, respectively, the summation over integral D such that C and D form a symmetric pair modulo the equivalence relation $\equiv \bmod C$, and the subsummation over such D restricted to relatively prime pairs.

Let h be a complex valued function on $(\mathbf{Z}\backslash\mathbf{Q})^3$. We will identify the space of 2×2 symmetric rational matrices with $\mathbf{Q}^3$ so that the symmetric matrix $X = \begin{pmatrix} x_4 & x_3 \\ x_3 & x_1 \end{pmatrix}$ corresponds to the element (x_1, x_3, x_4) of $\mathbf{Q}^3$. Thus h is a function on the space of rational symmetric matrices. For a nonsingular integral C, let

$$S_h^*(C) = \sum_{\substack{D \bmod C \\ \gcd(D,C)=1}} h(C^{-1}D), \qquad S_h(C) = \sum_{D \bmod C} h(C^{-1}D).$$

These make sense since if C, D are a symmetric pair, then $C^{-1}D$ is symmetric. It is also easy to see that for any $U \in GL(2, \mathbf{Z})$, $S_h(UC) = S_h(C)$ and $S_h^*(UC) = S_h^*(C)$.

Now arguing from Propositions 4.1 and 4.2 just as in the proof of the classical Möbius inversion formula, we have

Proposition 4.3. $S_h^*(C) = \sum_{H|C} \mu(H) S_h(H^{-1}C)$. □

In the sequel, we need as well a version of Proposition 4.3 which incorporates certain congruence conditions. Let N be a positive integer, let h be as above, and let C be a nonsingular integral matrix with entries divisible by N. Define

$$S_{h,N}^*(C) = \sum_{\substack{D \bmod C \\ \gcd(D,C)=1 \\ D_{12} \equiv 0 \bmod N}} h(C^{-1}D), \qquad S_{h,N}(C) = \sum_{\substack{D \bmod C \\ \gcd(\det D, N)=1 \\ D_{12} \equiv 0 \bmod N}} h(C^{-1}D).$$

Then

Proposition 4.4. *We have*

$$S_{h,N}^*(C) = \sum_{\substack{H|C \\ \gcd(\det H, N)=1}} \mu(H) S_{h,N}(H^{-1}C).$$

Proof. Observe that in the sum $S_{h,N}^*(C)$, one always has $\gcd(\det D, N) = 1$ since $C \equiv 0 \bmod N$ while $\gcd(C, D) = 1$. Then arguing as in the classical Möbius inversion formula, the Proposition follows. □

5. The Fourier coefficients of the Eisenstein series

In this section we study the Fourier coefficients of the Eisenstein series $E_s(g, W)$. We will follow the notation of section 2 with $\Phi_j = E_s|J^j$, where coefficients $B_j(g; T, R)$ are defined. The results of this section are in fact valid for the Eisenstein series formed with any function I satisfying (1.15).

Let us evaluate $B_1(g; T, R)$. By definition,

$B_j(g; T, R) =$

$$N^{-3j} \int_{(\mathbf{R}/N^j\mathbf{Z})^3} \int_{(\mathbf{R}/\mathbf{Z})^2} \Phi_j\left(\begin{pmatrix} E & X \\ & E \end{pmatrix} g, W\right) e(-N^{-j}\mathrm{tr}(T(Z_g + X)) - N^{1-j}\,{}^{\mathrm{T}}RW)\,dW\,dX. \tag{5.1}$$

Given a nonsingular integral matrix C, half integral symmetric T, and $R \in \mathbf{Z}^2$, let $S_1(C; T, R) =$

$$\sum_{\substack{D \bmod NC \\ C^{\mathsf{T}} D = D^{\mathsf{T}} C \\ \gcd(D, C) = 1 \\ D \equiv 0 \bmod N}} \sum_{\lambda \in \mathbf{Z}^2 / {}^{\mathsf{T}} C \mathbf{Z}^2} e(- {}^{\mathsf{T}} R C^{-1} D \lambda + m (C^{-1} D)[\lambda] + N^{-1} \operatorname{tr}(T C^{-1} D)) \,. \quad (5.2)$$

Here the notation $\gcd(D, C) = 1$ is explained in Section 4. Also, if $Y = Q^{\mathsf{T}} Q$ is a positive definite real symmetric matrix, define

$H(Q, s; C, T, R) =$

$$\frac{1}{2mN^3} \int_{\mathbf{R}^3} \sqrt{-\det(Z)} \left(\frac{\det(Y)}{|\det(Z)|^2} \right)^{s/2} I\left(\begin{pmatrix} 0 & -{}^{\mathsf{T}} C^{-1} \\ C & 0 \end{pmatrix} \begin{pmatrix} Q & X {}^{\mathsf{T}} Q^{-1} \\ 0 & {}^{\mathsf{T}} Q^{-1} \end{pmatrix} \right)$$

$$\times e\left(\frac{1}{4m} Z[R] - N^{-1} \operatorname{tr}(TZ) \right) dX \,,$$

where we are identifying a symmetric matrix $X = \begin{pmatrix} x_4 & x_3 \\ x_3 & x_1 \end{pmatrix}$ with the element (x_1, x_3, x_4) of $\mathbf{R}^3$, and $Z = X + iY$. Note that $H(Q, s; C, T, R)$ takes values in $\mathbf{V}$. Also, if $T = \frac{1}{4m}(U + N^{2-j} R\, {}^{\mathsf{T}} R)$, then

$$H(Q, s; C, N^{1-j} T, N^{1-j} R) = H\left(Q, s; C, \frac{N^{1-j}}{4m} U, 0 \right) .$$

Proposition 5.1. *For* re(s) *sufficiently large and*

$$g = \begin{pmatrix} Q & X {}^{\mathsf{T}} Q^{-1} \\ & {}^{\mathsf{T}} Q^{-1} \end{pmatrix}$$

we have

$$B_1(g; T, R) = \sum_C S_1(C; T, R) |\det(C)|^{-s} H(Q, s; C, T, R) \,, \quad (5.3)$$

where the summation is over nonsingular integral C such that $C_{12} \equiv 0 \bmod N$, modulo the left action of $\Gamma^0(N)$.

Proof. We may rewrite the sum giving $\Phi_1 = E_s | J$ as

$\Phi_1(g, W) =$

$$\sum_{\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in P \cap \Gamma \backslash \Gamma J} I_s(\gamma g) e^m(\gamma(Z)[\lambda] + 2 {}^{\mathsf{T}} W (CZ + D)^{-1} \lambda - ((CZ + D)^{-1} C)[W]) \,,$$

where $Z = g(iE) = X_g + iY_g$. Here since $D \equiv 0 \bmod N$, and $N > 1$, C must be nonsingular. Now

$$P \cap \Gamma = \left\{ \begin{pmatrix} {}^{\mathsf{T}} U^{-1} & \\ & U \end{pmatrix} \begin{pmatrix} E & S \\ & E \end{pmatrix} \middle| U \in \Gamma^0(N), S = {}^{\mathsf{T}} S \in M(2, \mathbf{Z}) \right\} .$$

Hence two integral symplectic matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \qquad \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$$

are $P \cap \Gamma$-equivalent if and only if

$$C = UC', \qquad D = UD', \qquad U \in \Gamma^0(N)\,.$$

Moreover,

$$\left\{ \begin{pmatrix} E & S \\ & E \end{pmatrix} \middle| S = {}^{\mathrm{T}}S \in M(2, \mathbf{Z}), S \equiv 0 \bmod N \right\}$$

acts properly on the *right* on $P \cap \Gamma \backslash \Gamma J$; this action takes the bottom row (C, D) to bottom row $(C, D + CS)$. Using this to unravel the integral in X, and translating by X_g, we have

$$B_1(g; T, R) = N^{-3} \sum_{\substack{C \in \Gamma^0(N)\backslash M(2,\mathbf{Z}) \\ C \text{ nonsingular} \\ C_{12} \equiv 0 \bmod N}} \sum_{\substack{D \bmod NC \\ C^{\mathrm{T}} D = D^{\mathrm{T}} C \\ \gcd(D, C) = 1 \\ D \equiv 0 \bmod N}} \sum_{\lambda \in \mathbf{Z}^2} \int_{\mathbf{R}^3} \int_{(\mathbf{R}/\mathbf{Z})^2} I_s\left(\gamma\begin{pmatrix} Q & X^{\mathrm{T}}Q^{-1} \\ & {}^{\mathrm{T}}Q^{-1} \end{pmatrix}\right) \times$$

$$e^m(\gamma(Z)[\lambda] + 2{}^{\mathrm{T}}W(CZ + D)^{-1}\lambda - ((CZ + D)^{-1}C)[W]) \times$$
$$e(-N^{-1}\mathrm{tr}(TZ) - {}^{\mathrm{T}}RW)\,dW\,dX\,,$$

with

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma J, \qquad Z = X + iY, \quad Y = Q^{\mathrm{T}}Q\,.$$

Since γ is symplectic and C is nonsingular, we may express the integrand as

$$I_s\left(\gamma\begin{pmatrix} Q & X^{\mathrm{T}}Q^{-1} \\ & {}^{\mathrm{T}}Q^{-1} \end{pmatrix}\right)$$
$$\times\, e^m(-(Z + C^{-1}D)^{-1}[W - C^{-1}\lambda] + (AC^{-1})[\lambda])e(-N^{-1}\mathrm{tr}(TZ) - {}^{\mathrm{T}}RW)\,.$$

From this expression it follows that translating λ by an element $C\lambda_1$ ($\lambda_1 \in \mathbf{Z}^2$) has the same effect as translating W by $-\lambda_1$. We use this to unravel the integral in W to obtain

$$B_1(g; T, R) =$$

$$N^{-3} \sum_{\substack{C \in \Gamma^0(N)\backslash M(2,\mathbf{Z}) \\ C \text{ nonsingular} \\ C_{12} \equiv 0 \bmod N}} \sum_{\substack{D \bmod NC \\ C^{\mathrm{T}} D = D^{\mathrm{T}} C \\ \gcd(D, C) = 1 \\ D \equiv 0 \bmod N}} \sum_{\lambda \in \mathbf{Z}^2/C\mathbf{Z}^2} e(-{}^{\mathrm{T}}RC^{-1}\lambda + m(AC^{-1})[\lambda]) \times$$

$$\int_{\mathbf{R}^5} I_s\left(\gamma\begin{pmatrix} Q & X^{\mathrm{T}}Q^{-1} \\ & {}^{\mathrm{T}}Q^{-1} \end{pmatrix}\right) e(-m(Z + C^{-1}D)^{-1}[W] - N^{-1}\mathrm{tr}(TZ) - {}^{\mathrm{T}}RW)\,dW\,dX\,.$$

Next we change variables in X, sending X to $X - C^{-1}D$. Evaluating the integral in W by means of the formula

$$\int_{\mathbf{R}^2} e^m(-Z^{-1}[W] - {}^{\mathrm{T}}RW)\,dW = \frac{\sqrt{-\det(Z)}}{2m}\, e\left(\frac{Z[R]}{4m}\right),$$

we have

$B_1(g; T, R) =$

$$\frac{1}{2mN^3} \sum_{\substack{C \in \Gamma^0(N)\backslash M(2,\mathbf{Z}) \\ C \text{ nonsingular} \\ C_{12} \equiv 0 \bmod N}} \sum_{\substack{D \bmod NC \\ C^{\mathsf{T}}D = D^{\mathsf{T}}C \\ \gcd(D,C)=1 \\ D \equiv 0 \bmod N}} \sum_{\lambda \in \mathbf{Z}^2/C\mathbf{Z}^2} e(-{}^{\mathsf{T}}RC^{-1}\lambda + m(AC^{-1})[\lambda]$$

$$+ N^{-1}\mathrm{tr}(TC^{-1}D)) \int_{\mathbf{R}^3} I_s\left(\begin{pmatrix} A & {}^{\mathsf{T}}C^{-1} \\ C & \end{pmatrix}\begin{pmatrix} Q & X\,{}^{\mathsf{T}}Q^{-1} \\ & {}^{\mathsf{T}}Q^{-1} \end{pmatrix}\right)\sqrt{-\det(Z)}\,e\left(\frac{Z[R]}{4m}\right)$$

$$e(-N^{-1}\mathrm{tr}(TZ))\,dX\ .$$

Given $\lambda, \lambda' \in \mathbf{Z}^2$, there exist $\lambda_1, \lambda_1' \in \mathbf{Z}^2$ such that $\lambda \equiv D\lambda_1$, $\lambda' \equiv D\lambda_1' \bmod C\mathbf{Z}^2$, since $D^{\mathsf{T}}A - C^{\mathsf{T}}B = E$. If $\lambda \equiv \lambda' \bmod C\mathbf{Z}^2$, then $\lambda_1 \equiv \lambda_1' \bmod {}^{\mathsf{T}}C\mathbf{Z}^2$, and conversely. Thus we may replace λ by $D\lambda$ in the sum and $\mathbf{Z}^2/C\mathbf{Z}^2$ by $\mathbf{Z}^2/{}^{\mathsf{T}}C\mathbf{Z}^2$. If we do so, the exponential terms which pull out of the integrand sum to $S_1(C; T, R)$. Since from (1.15) and (1.17) it follows that

$$I_s\left(\begin{pmatrix} A & -{}^{\mathsf{T}}C^{-1} \\ C & \end{pmatrix} g\right) = |\det(C)|^{-s}\left(\frac{\det(Y)}{|\det(Z)|^2}\right)^{s/2} I\left(\begin{pmatrix} & {}^{\mathsf{T}}C^{-1} \\ C & \end{pmatrix} g\right),$$

the Proposition follows. □

Next we turn to the evaluation of $B_0(g; T, R)$. Given a nonsingular integral matrix $C \equiv 0 \bmod N$, half integral symmetric T, and $R \in \mathbf{Z}^2$, let

$S_0(C; T, R) =$

$$\sum_{\substack{D \bmod NC \\ C^{\mathsf{T}}D = D^{\mathsf{T}}C \\ \gcd(D,C)=1 \\ D_{12} \equiv 0 \bmod N}} \sum_{\lambda \in \mathbf{Z}^2/{}^{\mathsf{T}}C\mathbf{Z}^2} e(-N\,{}^{\mathsf{T}}RC^{-1}D\lambda + m(C^{-1}D)[\lambda] + \mathrm{tr}(TC^{-1}D))\ .$$

The evaluation of $B_0(g; T, R)$ is more complicated than that of $B_1(g; T, R)$, due to the contribution to B_0 from singular C. For our purposes, the following result is sufficient.

Proposition 5.2. *For* $\mathrm{re}(s)$ *sufficiently large, and* $g = \begin{pmatrix} Q & X\,{}^{\mathsf{T}}Q^{-1} \\ & {}^{\mathsf{T}}Q^{-1} \end{pmatrix}$ *with* $Y_g = Q\,{}^{\mathsf{T}}Q$, *we have*

$$B_0(g; T, R) = \sum_C S_0(C; T, R)|\det(C)|^{-s}N^3 H(Q, s; C, NT, NR)$$

$$+ \delta_0'(\det Y_g)^{s/2} \int_{(\mathbf{R}/\mathbf{Z})^3} \left[I(g) + I\left(\begin{pmatrix} \eta & \\ & \eta \end{pmatrix} g\right)\right] e\left(\frac{N^2}{4m}Z[R] - \mathrm{tr}(TZ)\right)dX + S_{\text{“rank one”}}\,,$$

where the summation is over nonsingular integral C *such that* $C \equiv 0 \bmod N$, *modulo the left action of* $\Gamma^0(N)$; *where*

$$\delta_0' = \begin{cases} 1 & \text{if } \dfrac{N}{2m}R \in \mathbf{Z}^2\ , \\ 0 & \text{otherwise}\ ; \end{cases}$$

and where $S_{\text{“rank one”}}$ is the contribution from the rank one matrices C:

$$S_{\text{“rank one”}} = \int_{(\mathbf{R}/\mathbf{Z})^5} \sum_{\substack{\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in P \cap \Gamma \backslash \Gamma \\ \operatorname{rank}(C) = 1 \\ \lambda \in \mathbf{Z}^2}} \left(I_s \| \begin{bmatrix} \lambda \\ 0 \end{bmatrix} | \gamma \right)(g, W) e(-\operatorname{tr}(TZ) - N\,{}^{T}RW) dW dX.$$

In these integrals, Z denotes $X + iY_g$.

Proof. Substituting the expression for $E_s(g, W)$ into the integral (5.1), we see that the integral breaks up into three pieces, according to the rank of C, which may be 0, 1 or 2. The rank $(C) = 0$ term is immediately seen to be

$$\int_{(\mathbf{R}/\mathbf{Z})^5} \sum_{\lambda \in \mathbf{Z}^2} \left[I_s(g) + I_s\left(\begin{pmatrix} \eta & \\ & \eta \end{pmatrix} g \right) \right] e^m(Z[\lambda] + 2^{T} W \lambda) e(-\operatorname{tr}(TZ) - N\,{}^{T}RW) dW dX .$$

Since the W integral is 1 when $\lambda = \dfrac{N}{2m} R$ and 0 otherwise, we obtain the second term in Proposition 5.2. The rank$(C) = 1$ term is of course $S_{\text{“rank one”}}$. Finally, when C is nonsingular, the computation is similar to that in Proposition 5.1. Note that in this case $\left\{ \begin{pmatrix} E & S \\ 0 & E \end{pmatrix} \middle| S = {}^{T}S \in M(2, \mathbf{Z}) \right\}$ acts properly on $P \cap \Gamma \backslash \Gamma$, and we sum over $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in $P \cap \Gamma \backslash \Gamma$ rather than in $P \cap \Gamma \backslash \Gamma J$; these account for the differences between S_0 and S_1. We omit the details. □

6. The Whittaker coefficients of the Eisenstein series

We continue to analyze the coefficients of $\Phi_j = E_s | J^j$, this time with special choices of R and T. The integrals which we consider in this section will later be reinterpreted as *Whittaker coefficients* of the Eisenstein series $\mathscr{E}_j$ of half integral weight.

Suppose that $v = {}^{T}(0, r)$, with $r \in \mathbf{Z}$, $\Delta \in \mathbf{Z}$ and $0 < n_2 \in N^{-1}\mathbf{Z}$. It follows from Corollary 2.8 that the following integral is well-defined.

$$\mathscr{C}_1(s; \Delta, n_2, r; y_1, y_2) = N^{-1} \int_{\mathbf{R}/N\mathbf{Z}} C_1$$
$$\left(\begin{pmatrix} 1 & x_2 & & \\ & 1 & & \\ & & 1 & \\ & & -x_2 & 1 \end{pmatrix} \begin{pmatrix} y_1^{1/2} y_2 & & & \\ & y_1^{1/2} & & \\ & & y_1^{-1/2} y_2^{-1} & \\ & & & y_1^{-1/2} \end{pmatrix}; U_1(-N\Delta), v \right) e(-n_2 x_2) dx_2 .$$

According to (2.6), this expression vanishes unless $n_1 = \dfrac{N}{4m}(r^2 - \Delta)$ is an integer, and we assume this. Given such r, Δ and n_2, let us define a Dirichlet series

$$\mathscr{L}(s, \Delta, n_2) = \sum_{\substack{\alpha, \beta, \delta \in \mathbf{Z} \\ \alpha, \delta > 0 \\ \beta \bmod N\delta \\ \beta \equiv 0 \bmod N \\ \alpha | N n_2 \delta}} S_1\left(\begin{pmatrix} \alpha & \beta \\ & \delta \end{pmatrix}; U_1(n_1), v \right) (\alpha\delta)^{-s} \left(\frac{\alpha}{\delta} \right)^{k/2} a\left(\frac{N n_2 \delta}{\alpha} \right) e\left(\frac{n_2 \beta}{\alpha} \right),$$

where $S_1(C; T, R)$ is defined in Section 5. If $n_2 = N^{-1}$, we will denote $\mathscr{L}(s, \Delta) = \mathscr{L}(s, \Delta, N^{-1})$.

Proposition 6.1. *We have*

$\mathscr{C}_1(s; \Delta, n_2, r; y_1, y_2) =$

$$\begin{cases} \left(\frac{n_2|\Delta|}{4m}\right)^{s-4} n_2^{-k/2} e\left(\frac{iy_1\Delta}{4m}\right)\mathscr{L}(s, \Delta, n_2) W^+\left(\frac{|\Delta|}{4m}y_1, n_2 y_2; s\right) & \text{if } \Delta > 0; \\ \left(\frac{n_2|\Delta|}{4m}\right)^{s-4} n_2^{-k/2} e\left(\frac{iy_1\Delta}{4m}\right)\mathscr{L}(s, \Delta, n_2) W^-\left(\frac{|\Delta|}{4m}y_1, n_2 y_2; s\right) & \text{if } \Delta < 0; \\ n_2^{s-4-k/2}\mathscr{L}(s, \Delta, n_2) W^0(y_1, n_2 y_2; s) & \text{if } \Delta = 0\,. \end{cases}$$

Proof. In the notation of section 4, we must evaluate

$$\frac{1}{lN}\int_{\mathbf{R}/lN\mathbf{Z}} H\left(Q, s; C, \frac{1}{4m}U_1(-N\Delta), 0\right) e(-n_2 x_2)\, dx_2 \tag{6.1}$$

with $Q = \sqrt{y_1}\begin{pmatrix} y_2 & x_2 \\ & 1\end{pmatrix}$, where l is chosen sufficiently large that the integral is well defined. Observe that

$$\begin{aligned}\begin{pmatrix} & -{}^{\mathrm{T}}C^{-1} \\ C & \end{pmatrix}\begin{pmatrix} Q & X\,{}^{\mathrm{T}}Q^{-1} \\ & {}^{\mathrm{T}}Q^{-1}\end{pmatrix} &= \begin{pmatrix} {}^{\mathrm{T}}C^{-1}w & \\ & Cw\end{pmatrix}\begin{pmatrix} & w \\ -w & \end{pmatrix}\begin{pmatrix} Q & X\,{}^{\mathrm{T}}Q^{-1} \\ & {}^{\mathrm{T}}Q^{-1}\end{pmatrix} \\ &= \begin{pmatrix} {}^{\mathrm{T}}C^{-1}w & \\ & Cw\end{pmatrix}\begin{pmatrix} E(x_2) & \\ & {}^{\mathrm{T}}E(x_2)^{-1}\end{pmatrix}\begin{pmatrix} & w \\ -w & \end{pmatrix}\begin{pmatrix} Y & \bar{X}\,{}^{\mathrm{T}}Y^{-1} \\ & {}^{\mathrm{T}}Y^{-1}\end{pmatrix},\end{aligned}$$

where $Y = \sqrt{y_1}\begin{pmatrix} y_2 & \\ & 1\end{pmatrix}$, $w = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$ and $\bar{X}$ denotes $E(-x_2)X\,{}^{\mathrm{T}}E(-x_2)$. Change X to $E(x_2)X\,{}^{\mathrm{T}}E(x_2)$ in the integral. Note that the quantity $e((4m)^{-1}\operatorname{tr}(U_1(\Delta)Z))$ is unchanged by this substitution, while $\det(Z)$ becomes

$$\det(E(x_2)X\,{}^{\mathrm{T}}E(x_2) + iE(x_2)Y^{\mathrm{T}}Y\,{}^{\mathrm{T}}E(x_2)) = \det(X + iY^{\mathrm{T}}Y)\,,$$

which is independent of x_2. Then using (1.14), (1.16) and (3.51), the integral (6.1) becomes

$$\begin{aligned}&\frac{1}{2mlN^4}\int_{\mathbf{R}/lN\mathbf{Z}}\int_{\mathbf{R}^3}[-\det(X + iY^{\mathrm{T}}Y)]^{1/2} e\left(\frac{\Delta}{4m}(x_1 + iy_1)\right) \\ &\quad\times\left[\frac{y_1 y_2}{\det(X + iY^{\mathrm{T}}Y)}\right]^s F({}^{\mathrm{T}}C^{-1}wE(x_2)Q'\varepsilon_C)e(-n_2x_2)\cdot\mathrm{v}\sigma(\varepsilon_C\kappa)\,dX\,dx_2\end{aligned} \tag{6.2}$$

where

$$X = \begin{pmatrix} x_4 & x_3 \\ x_3 & x_1\end{pmatrix}, \qquad \varepsilon_C = \begin{pmatrix} 1 & 0 \\ 0 & \operatorname{sgn}(\det C)\end{pmatrix},$$

and $Q' = \sqrt{y_1'}\begin{pmatrix} y_2' & x_2' \\ & 1 \end{pmatrix}$ is as in (3.51). Let us evaluate the integral

$$\frac{1}{lN}\int_{\mathbf{R}/lN\mathbf{Z}} F\left({}^{\mathrm{T}}C^{-1}wE(x_2)Q'\varepsilon_C\right)e(-n_2x_2)\,dx_2 .$$

Now C is to be taken modulo $\Gamma^0(N)$. Hence we must evaluate the integral for C of the form $C = \varrho\varepsilon_C\begin{pmatrix} \alpha & \beta \\ & \delta \end{pmatrix}$ where $\alpha, \beta, \delta \in \mathbf{Z}$, with $\alpha, \delta > 0$, where β is taken modulo δ, and ϱ ranges over a set of coset representatives for $\Gamma^0(N)\backslash SL(2,\mathbf{Z})$. These representatives correspond to positive integers a, b such that $a|N$, $\gcd(a,b)=1$, and b is taken modulo $a^{-1}N\mathbf{Z}$; the corresponding representative is given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with c and d chosen so that $ad - bc = 1$. However, we are only concerned with C such that $C_{12} \equiv 0 \bmod N$ and $S_1(C; U_1(n_1), v) \neq 0$. Since then there exists a $D \equiv 0 \bmod N$ such that $\gcd(C,D)=1$, we must have $\gcd(C_{11}, N) = 1$, and hence $a = 1$. This means that ϱ is of the form $\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}$, $b \in \mathbf{Z}/N\mathbf{Z}$; we conclude that we must evaluate the integral when

$$C = \varepsilon_C\begin{pmatrix} \alpha & \beta \\ & \delta \end{pmatrix}, \qquad \alpha, \delta > 0,\ \beta \bmod N\delta,\ \beta \equiv 0 \bmod N .$$

Then

$${}^{\mathrm{T}}C^{-1}w = \varepsilon_C w\begin{pmatrix} \delta^{-1} & \beta\alpha^{-1}\delta^{-1} \\ & \alpha^{-1} \end{pmatrix}.$$

Since $n_2 > 0$, we get for the x_2 integral,

$$\frac{1}{lN}\int_{\mathbf{R}/lN\mathbf{Z}} F\left(\varepsilon_C w\begin{pmatrix} \delta^{-1} & \beta\alpha^{-1}\delta^{-1} \\ & \alpha^{-1} \end{pmatrix}E(x_2)Q'\varepsilon_C\right)e(-n_2x_2)\,dx_2$$
$$= \begin{cases} \left(\dfrac{\alpha}{\delta}\right)^{k/2} e\left(\dfrac{n_2\beta}{\alpha}\right)a\left(\dfrac{Nn_2\delta}{\alpha}\right)e(n_2(x_2' + iy_2'))(y_2')^{k/2} & \text{if } N\delta n_2/\alpha \in \mathbf{Z},\ \det C > 0; \\ 0 & \text{otherwise} . \end{cases}$$

Substituting this expression into (6.2), using (3.52) and (3.53), and summing over C as above, we obtain the formula of Proposition 6.1. □

We continue to assume that $v = {}^{\mathrm{T}}(0, r)$ with $r \in \mathbf{Z}$. It follows from Corollary 2.8 that the following integral is well-defined.

$$\mathscr{C}_0(s; \Delta, n_2, r; y_1, y_2) = N^{-1}\int_{\mathbf{R}/N\mathbf{Z}} \sum_{\mu \bmod 2m/N} e\left(-\frac{N}{2m}{}^{\mathrm{T}}v\mu\right)$$
$$C_0\left(\left(\begin{pmatrix} 1 & & & \\ -x_2 & 1 & & \\ & & 1 & x_2 \\ & & & 1 \end{pmatrix}\begin{pmatrix} y_1^{1/2} & & & \\ & y_1^{1/2}y_2 & & \\ & & y_1^{-1/2} & \\ & & & y_1^{-1/2}y_2^{-1} \end{pmatrix}; U_0(-\Delta), \mu\right)e(-n_2x_2)\,dx_2 .$$

Again by (2.6), this vanishes unless $\Delta \equiv 0 \bmod 4m$, and we now assume this. Given such r, Δ and $0 < n_2 \in N^{-1}\mathbf{Z}$, define the Dirichlet series

$$\hat{\mathscr{L}}(s, \Delta, n_2, r) = \sum_{\mu \bmod 2m/N} e\left(-\frac{N}{2m}{}^{\mathrm{T}}v\mu\right) \sum_{\gamma \in \Gamma_0(N)\backslash SL(2,\mathbf{Z})} \sum_{\substack{\alpha,\beta,\delta\in\mathbf{Z}\\ \alpha,\delta>0\\ \beta \bmod \delta\\ \alpha | N n_2\delta}} S_0\left(N^{\mathrm{T}}\gamma^{-1}\begin{pmatrix}\alpha & \beta\\ & \delta\end{pmatrix} w, T, \mu\right) \times (N^2\alpha\delta)^{-s}\left(\frac{\alpha}{\delta}\right)^{k/2} a_\gamma\left(\frac{Nn_2\delta}{\alpha}\right) e\left(\frac{n_2\beta}{\alpha}\right),$$

where $T = \frac{1}{4m}(U_0(-\Delta) + N^2\mu^{\mathrm{T}}\mu)$, and $S_0(C; T, \mu)$ is defined in Section 5. As with $\mathscr{L}$, we abbreviate $\hat{\mathscr{L}}(s, \Delta, N^{-1}, r)$ as $\hat{\mathscr{L}}(s, \Delta, r)$.

Proposition 6.2. *We have*

$$\mathscr{C}_0(s; \Delta, n_2, r; y_1, y_2) = \begin{cases} N^3\left(\frac{n_2|\Delta|}{4m}\right)^{s-4} n_2^{-k/2} e\left(\frac{iy_1\Delta}{4m}\right)\hat{\mathscr{L}}(s,\Delta,n_2,r)W^+\left(\frac{|\Delta|}{4m}y_1,\ n_2y_2; s\right)\sigma(w) & \text{if } \Delta > 0;\\ N^3\left(\frac{n_2|\Delta|}{4m}\right)^{s-4} n_2^{-k/2} e\left(\frac{iy_1\Delta}{4m}\right)\hat{\mathscr{L}}(s,\Delta,n_2,r)W^-\left(\frac{|\Delta|}{4m}y_1,\ n_2y_2; s\right)\sigma(w) & \text{if } \Delta < 0;\\ N^3 n_2^{s-k/2-4}\hat{\mathscr{L}}(s,\Delta,n_2,r)W^0(y_1,\ n_2y_2; s)\sigma(w) & \\ \quad + (y_1y_2)^s y_2^{k/2} a(Nn_2)e(n_2iy_2)\mathbf{v}\sigma(\eta) & \text{if } \Delta = 0.\end{cases}$$

Proof. The evaluation of the integral breaks up into the three pieces of Proposition 5.2. The contribution from the rank two term is treated by a method similar to that of Proposition 6.1. One first shows that if $Q'' = \sqrt{y_1}\begin{pmatrix} & 1\\ -x_2y_2 & \end{pmatrix}$, then

$$(lN)^{-1}\int_{\mathbf{R}/lN\mathbf{Z}} H(Q'', s; C, (4m)^{-1}NU_0(-\Delta), 0)e(-n_2x_2)dx_2 = \frac{1}{2mN^3}\int_{\mathbf{R}^3}\sqrt{-\det(X+iY^{\mathrm{T}}Y)}\,e\left(\frac{\Delta}{4m}(x_1+iy_1)\right)\left[\frac{y_1y_2}{\det(X+iY^{\mathrm{T}}Y)}\right]^s \times \left\{(lN)^{-1}\int_{\mathbf{R}/lN\mathbf{Z}} F({}^{\mathrm{T}}C^{-1}E(x_2)Q'\varepsilon_C)e(-n_2x_2)dx_2\right\}\mathbf{v}\sigma(\varepsilon_C\kappa w)dX,$$

where l is chosen sufficiently large that the integral is well defined. The integral in x_2 is evaluated by means of (1.19); the integral in X gives rise to the Whittaker function W. As for the rank zero term, it follows from Proposition 4.2 that only the summand with $\mu = 0$ contributes. Using (1.14), (1.16) and (1.19), one arrives at

$$(y_1y_2)^s y_2^{k/2} a(Nn_2)e(n_2iy_2)\int_{(\mathbf{R}/\mathbf{Z})^3} e((4m)^{-1}\mathrm{tr}(U_0(\Delta)Z))dX\,\mathbf{v}\sigma(\eta).$$

This equals 0 unless $\Delta = 0$, in which case it equals

$$(y_1y_2)^s y_2^{k/2} a(Nn_2)e(n_2iy_2)\mathbf{v}\sigma(\eta).$$

Hence to complete the proof, we must show that

$$N^{-1} \int_{\mathbf{R}/N\mathbf{Z}} S_{\text{"rank one"}}\, e(-n_2 x_2)\, dx_2 = 0\,.$$

In fact, we will show that the contribution of each individual γ to this integral is zero. This is accomplished as follows.

Since C is of rank one, it follows from Maass [14], page 160 that we may write

$$\gamma = \begin{pmatrix} {}^{\mathrm{T}}V_1^{-1} & \\ & V_1 \end{pmatrix} \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \begin{pmatrix} {}^{\mathrm{T}}V_2^{-1} & \\ & V_2 \end{pmatrix}$$

with V_1 and V_2 integral, $\det(V_1) = |\det(V_2)| = 1$, and

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}, \quad C_1 = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}, \quad D_1 = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix},$$

with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z})$, and $c \neq 0$. It is sufficient to show that the contributions with $V_2 = E$ give zero, since the general case may be reduced to this by the variable changes $X \to {}^{\mathrm{T}}V_2 X V_2$ and $W \to {}^{\mathrm{T}}V_2^{-1} W$. Then

$$\begin{aligned} &e^m(\gamma(Z)[\lambda] + 2{}^{\mathrm{T}}W(CZ + D)^{-1}\lambda - ((CZ + D)^{-1}C)[W]) \\ &\quad = e^m\Bigg(-((C_1 Z + D_1)^{-1} C_1)\left[W - \begin{pmatrix} 0 & \\ & c^{-1} \end{pmatrix}\lambda'\right] + \left(A_1 \begin{pmatrix} 0 & \\ & c^{-1} \end{pmatrix}\right)\left[\begin{pmatrix} 0 \\ \lambda_2' \end{pmatrix}\right] \\ &\qquad + 2{}^{\mathrm{T}}W(CZ + D)^{-1}\begin{pmatrix} \lambda_1 \\ 0 \end{pmatrix} + \gamma(Z)\left[\begin{pmatrix} \lambda_1 \\ 0 \end{pmatrix}\right] + 2(\lambda_1, 0)\gamma(Z)\begin{pmatrix} 0 \\ \lambda_2 \end{pmatrix}\Bigg), \end{aligned}$$

where $\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$ and $\lambda' = V_1^{-1}\lambda = \begin{pmatrix} \lambda_1' \\ \lambda_2' \end{pmatrix}$. As in the proof of Proposition 5.1, it follows that translating λ by $V_1\begin{pmatrix} 0 & \\ & c \end{pmatrix}\begin{pmatrix} 0 \\ t \end{pmatrix}$, for $t \in \mathbf{Z}$, has the same effect as translating W by $-\begin{pmatrix} 0 \\ t \end{pmatrix}$. We use this to unravel the integral in w_2, where $W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$. Then, by changing variables $W \to W + \begin{pmatrix} 0 & \\ & c^{-1} \end{pmatrix}\lambda'$, we obtain

$$\sum_{\lambda \in \mathbf{Z}^2/C\mathbf{Z}^2} e\left(mA_1 \begin{pmatrix} 0 & \\ & c^{-1} \end{pmatrix}\left[\begin{pmatrix} 0 \\ \lambda_2' \end{pmatrix}\right] - N{}^{\mathrm{T}}\mu \begin{pmatrix} 0 & \\ & c^{-1} \end{pmatrix}\lambda' \right)$$

$$\begin{aligned} &(Nl)^{-1} \int_{\mathbf{R}/Nl\mathbf{Z}} \int_{\mathbf{R}} \int_{(\mathbf{R}/\mathbf{Z})^4} I_s(\gamma g) \\ &\quad \times e^m\left(-((C_1 Z + D_1)^{-1} C_1)[W] + 2{}^{\mathrm{T}}W(CZ + D)^{-1}\begin{pmatrix} \lambda_1 \\ 0 \end{pmatrix} + \gamma(Z)\left[\begin{pmatrix} \lambda_1 \\ 0 \end{pmatrix}\right]\right) \\ &\quad \times e(-\operatorname{tr}(TZ) - N{}^{\mathrm{T}}\mu W - n_2 x_2)\, dX\, dw_1\, dw_2\, dx_2\,. \end{aligned}$$

Now we may evaluate the w_1 and w_2 integrals to obtain

$$\sum_{\substack{\lambda\in \mathbf{Z}^2/C\mathbf{Z}^2 \\ \lambda_1' = \mu_1 N/2m}} e\left(mA_1\begin{pmatrix}0 & \\ & c^{-1}\end{pmatrix}\left[\begin{pmatrix}0\\ \lambda_2'\end{pmatrix}\right] - N\,{}^{\mathrm{T}}\mu\begin{pmatrix}0 & \\ & c^{-1}\end{pmatrix}\lambda' + \frac{dN^2}{4mc}\mu_2^2\right)$$

$$\times (Nl)^{-1} \int_{\mathbf{R}/Nl\mathbf{Z}}\int_{(\mathbf{R}/\mathbf{Z})^3} I_s(\gamma g)\sqrt{\frac{x_1 + iy_1|z_2|^2 + d/c}{2im}} \times$$

$$e((\Delta/4m)(x_4 + iy_1) - n_2x_2)dX\,dx_2\,, \quad (6.3)$$

with $X = \begin{pmatrix}x_4 & x_3\\ x_3 & x_1\end{pmatrix}$, $z_2 = x_2 + iy_2$. The essential feature is that in this integral, only $I_s(\gamma g)$ depends on x_3. We have the following Bruhat decomposition for γ:

$$\gamma = \begin{pmatrix}{}^{\mathrm{T}}V_1^{-1} & \\ & V_1\end{pmatrix}$$

$$\begin{pmatrix}1 & & & \\ & 1 & a/c & \\ & & 1 & \\ & & & 1\end{pmatrix}\begin{pmatrix}1 & & & \\ & 1/c & & \\ & & 1 & \\ & & & c\end{pmatrix}\begin{pmatrix}1 & & & \\ & & & -1\\ & & 1 & \\ & 1 & & \end{pmatrix}\begin{pmatrix}1 & & & \\ & 1 & d/c & \\ & & 1 & \\ & & & 1\end{pmatrix}.$$

Since

$$\begin{pmatrix}1 & & & \\ & & & -1\\ & & 1 & \\ & 1 & & \end{pmatrix}\begin{pmatrix}1 & & & x_3\\ & 1 & x_3 & \\ & & 1 & \\ & & & 1\end{pmatrix} = \begin{pmatrix}1 & -x_3 & & \\ & 1 & & \\ & & 1 & \\ & & x_3 & 1\end{pmatrix}\begin{pmatrix}1 & & & \\ & & & -1\\ & & 1 & \\ & 1 & & \end{pmatrix},$$

and since F is cuspidal, we conclude that the integral (6.3) vanishes. This completes the proof of Proposition 6.2. □

7. Evaluation of the Dirichlet series

The purpose of this section is to evaluate the Dirichlet series $\mathscr{L}(s, D) = \mathscr{L}(s, D, N^{-1})$ defined in Section 6.

Proposition 7.1. *If $D = D_0D_1^2$, where D_0 is a fundamental discriminant, then*

$$\mathscr{L}(s, D) = d(s, D_1)\prod_{p\nmid ND_1}\frac{(1-\chi_{D_0}(p)\sigma_p p^{2-k/2-s})^{-1}(1-\chi_{D_0}(p)\sigma_p' p^{2-k/2-s})^{-1}}{(1-\sigma_p^2p^{4-k-2s})^{-1}(1-\sigma_p'^2p^{4-k-2s})^{-1}(1-p^{3-2s})^{-1}},$$

where $d(s, D_1)$ is a polynomial in p^{-s} for $p|D_1$, given by (7.35) *below, and $d(s, 1) = 1$. In particular*

$$\mathscr{L}(s, D_0) = \frac{L_N(s + k/2 - 2, f, \chi_{D_0})}{L_N(2s + k - 4, f, \vee^2)}.$$

When $D = 0$,

$$\mathscr{L}(s, 0) = \frac{L_N(2s + k - 5, f, \vee^2)}{L_N(2s + k - 4, f, \vee^2)}.$$

If $s \geqq 2$ is real then $\sum_{D_1=1}^{\infty} \mathscr{L}(s, D_0 D_1^2) D_1^{-2u}$ *converges for* $\mathrm{re}(u) > \frac{3}{4}$. *Furthermore, if* $\mathscr{L}(2, D_0) = 0$, *then* $\sum_{D_1=1}^{\infty} \mathscr{L}'(2, D_0 D_1^2) D_1^{-2u}$ *converges for all* $\mathrm{re}(u) > \frac{3}{4}$.

Remark. In fact it can be shown that $\mathscr{L}(s, D_0 D_1^2) = \mathscr{L}(s, D_0) b(s, D_1)$, where $b(s, D_1)$ is a finite Dirichlet polynomial, and that the $b(s, D_1)$ satisfy a recursive relation similar to that given in Theorem 8.1 of [3]. We will not elaborate on this here because it is not necessary for our main result, and because we will present an easier method of doing the necessary calculations in a later paper, based on a representation theoretic technique due to Casselman and Shalika.

Of the coefficients $\hat{\mathscr{L}}(s, D, r)$, all that we require is that they are of at most polynomial growth in D. This is of course well-known for the Whittaker coefficients of any automorphic form, so we do not have to compute them in this section. (It may be shown that $\hat{\mathscr{L}}(s, D, r)$ equals $\mathscr{L}(s, D)$ times a finite Dirichlet polynomial.) We also need to know that $\hat{\mathscr{L}}(s, 0, r)$ has analytic continuation to a neighborhood of $s = 2$. In fact, a nearly identical calculation to that given here shows that $\hat{\mathscr{L}}(s, 0, r)$ is equal to $\mathscr{L}(s, 0)$ times a rational function in p^{-s} where p runs through the primes dividing N and D. We omit this calculation, which uses the second version, Proposition 4.4 of the Möebius inversion formula. Actually, given the Theorem of Section 8 it is possible to see directly that $\hat{\mathscr{L}}(s, 0, r)$ is analytic at $s = 2$ since (for large u) all other terms in the formula are analytic.

Proof. Recall that

$$\mathscr{L}(s, D) = \left\{ \sum_{\substack{\alpha, \beta, \delta \in \mathbf{Z} \\ \alpha, \delta > 0 \\ \beta \bmod N\delta \\ \beta \equiv 0 \bmod N \\ \alpha | \delta}} S_1\left(\begin{pmatrix} \alpha & \beta \\ & \delta \end{pmatrix}; U_1(n_1), v\right)(\alpha\delta)^{-s} \left(\frac{\alpha}{\delta}\right)^{k/2} a\left(\frac{\delta}{\alpha}\right) e\left(\frac{\beta}{N\alpha}\right)\right\}, \tag{7.1}$$

where $n_1 = \dfrac{N}{4m}(r^2 - D)$ and $v = (0, r)$. Our first objective is to show that the inner sum S_1 is a multiplicative function of δ, and to write it as a product of simpler p-factors.

Let us consider the definition (5.2) of $S_1(C; U_1(n_1), v)$. The condition $\gcd(D, C) = 1$ in the definition (5.2) implies that $S_1(C, U_1(n_1), v) = 0$ unless $\gcd(\det C, N) = 1$. Now if $\gcd(\det C, N) = 1$, then summing over D modulo NC such that $D \equiv 0 \bmod N$ is equivalent to summing over D modulo C. Therefore

$$S_1(C; U_1(n_1), v) =$$

$$\sum_{\substack{D \bmod C \\ C^{\mathsf{T}} D = D^{\mathsf{T}} C \\ \gcd(D, C) = 1}} \sum_{\lambda \in \mathbf{Z}^2 / {}^{\mathsf{T}} C \mathbf{Z}^2} e(-{}^{\mathsf{T}} v C^{-1} D \lambda + m(C^{-1} D)[\lambda] + N^{-1} \mathrm{tr}(U_1(n_1) C^{-1} D)), \tag{7.2}$$

if $\gcd(\det C, N) = 1$, and zero otherwise.

Note that because $\gcd(\delta, N) = 1$, it follows that in (7.1) β is taken modulo δ, satisfying $\beta \equiv 0 \bmod N$. By the Möbius inversion formula Proposition 4.3,

$$S_1(C, U_1(n_1), v) = \sum_{H | C} S(H^{-1} C, U_1(n_1), v), \tag{7.3}$$

where

$S(C; U_1(n_1), v) =$

$$\sum_{\substack{D \bmod C \\ C^{\mathsf T}D = D^{\mathsf T}C}} \sum_{\lambda \in \mathbf{Z}^2/{}^{\mathsf T}C\mathbf{Z}^2} e(-{}^{\mathsf T}vC^{-1}D\lambda + m(C^{-1}D)[\lambda] + N^{-1}\mathrm{tr}(U_1(n_1)C^{-1}D)) \qquad (7.4)$$

is the sum S_1 with the restriction $\gcd(C, D) = 1$ removed. We will focus our attention now on the sum S. Write $\beta = u'\beta'$ where $\gcd(u', \alpha\delta) = 1$ and $p|\beta'$ implies that $p|\alpha\delta$. Factoring α, β', δ into primes we may write $\delta = \prod p_i^{d_i}$, $\beta' = \prod p_i^{b_i}$, $\alpha = \prod p^{a_i}$, with $a_i, b_i, d_i \geqq 0$. Let

$$\Sigma_1 = \{p_i | a_i \leqq d_i,\ a_i \leqq b_i\},$$
$$\Sigma_2 = \{p_i | a_i > d_i,\ a_i \leqq b_i\},$$
$$\Sigma_3 = \{p_i | a_i > b_i\},$$

and for $k = 1, 2, 3$ define

$$\delta_k = \prod_{p_i \in \Sigma_k} p_i^{d_i}, \qquad \beta_k = \prod_{p_i \in \Sigma_k} p_i^{b_i}, \qquad \alpha_k = \prod_{p_i \in \Sigma_k} p_i^{a_i}.$$

Thus $\delta = \delta_1\delta_2\delta_3$, $\beta' = \beta_1\beta_2\beta_3$ and $\alpha = \alpha_1\alpha_2\alpha_3$. Note that after left multiplication by an element of $\Gamma^0(N)$ if necessary, which does not change the value of $S(C; U_1(n_1), v)$, we may assume that $\beta_3|\delta_3$. We use this factorization of α, β, δ in the following

Lemma 7.2. *We may write*

$$C = \begin{pmatrix} \alpha & \beta \\ & \delta \end{pmatrix} = \gamma_1 \begin{pmatrix} \alpha_1\alpha_2\beta_3 & \\ & \delta\alpha_3/\beta_3 \end{pmatrix} \gamma_2$$

where $\gamma_1, \gamma_2 \in SL(2, \mathbf{Z})$ *and*

$$\gamma_2 = \begin{pmatrix} v\alpha_3/\beta_3 & v'\beta_1\beta_2/(\alpha_1\alpha_2) \\ l & q \end{pmatrix}$$

with

$$v\alpha_3/\beta_3 \equiv 1 \bmod \delta_1\delta_2\alpha_1\alpha_2, \qquad v'\beta_1\beta_2/(\alpha_1\alpha_2) \equiv -1 \bmod \delta_3\alpha_3,$$
$$v \equiv -u^{-1}\alpha_1\alpha_2/(\beta_1\beta_2) \bmod \delta_3, \qquad v' \equiv u\beta_3\alpha_3^{-1} \bmod \delta_1\delta_2,$$
$$l \equiv 1 \bmod \delta_3\alpha_3, \qquad q \equiv 0 \bmod \delta_3\alpha_3,$$
$$l \equiv 0 \bmod \delta_1\delta_2\alpha_1\alpha_2, \qquad q \equiv 1 \bmod \delta_1\delta_2\alpha_1\alpha_2.$$

Proof. The easy verification is left to the reader. □

We now wish to parametrize the residue classes D modulo C. Write $D = \gamma_1 D' {}^{\mathsf T}\gamma_2^{-1}$, with $D' = \begin{pmatrix} u & x \\ y & w \end{pmatrix}$. We require that $D^{\mathsf T}C$ be symmetric, so as

$$D^{\mathsf T}C = \gamma_1 \begin{pmatrix} u\alpha_1\alpha_2 & x\delta\alpha_3/\beta_3 \\ y\alpha_1\alpha_2\beta_3 & w\delta\alpha_3/\beta_3 \end{pmatrix} {}^{\mathsf T}\gamma_1,$$

we must write $x = x_0\alpha_2/\delta_2$, $y = x_0\delta_1\delta_3\alpha_3/(\beta_3^2\alpha_1)$. For S a symmetric matrix, write $S = \gamma_2^{-1}S'^{\mathrm{T}}\gamma_2^{-1}$, with $S' = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix}$. Then

$$D + CS = \gamma_1 D'^{\mathrm{T}}\gamma_2^{-1} + \gamma_1\begin{pmatrix}\alpha_1\alpha_2\beta_3 & \\ & \delta\alpha_3/\beta_3\end{pmatrix}S'^{\mathrm{T}}\gamma_2^{-1} =$$

$$\gamma_1\left[D' + \begin{pmatrix}\alpha_1\alpha_2\beta_3 & \\ & \delta\alpha_3/\beta_3\end{pmatrix}\begin{pmatrix} s_1 & s_2 \\ s_2 & s_3\end{pmatrix}\right]^{\mathrm{T}}\gamma_2^{-1}$$

and so it is easily seen that the different residue classes are parametrized by taking all w modulo $\delta\alpha_3/\beta_3$, u modulo $\alpha_1\alpha_2\beta_3$ and x_0 modulo $\delta_2\beta_3\alpha_1$ in $D = \gamma_1 D'^{\mathrm{T}}\gamma_2^{-1}$, with

$$D' = \begin{pmatrix} u & x_0\alpha_2/\delta_2 \\ x_0\delta_1\delta_3\alpha_3(\beta_3^2\alpha_1)^{-1} & w\end{pmatrix}.$$

In terms of C and D as described above, we now have

$$C^{-1}D = \gamma_2^{-1}\begin{pmatrix}\dfrac{u}{\alpha_1\alpha_2\beta_3} & \dfrac{x_0}{\alpha_1\beta_3\delta_2} \\ \dfrac{x_0}{\alpha_1\beta_3\delta_2} & \dfrac{w}{\delta\alpha_3/\beta_3}\end{pmatrix}^{\mathrm{T}}\gamma_2^{-1}.$$

Let $\lambda' = \begin{pmatrix}\lambda_1' \\ \lambda_2'\end{pmatrix} = {}^{\mathrm{T}}\gamma_2^{-1}\lambda$, so ${}^{\mathrm{T}}\lambda\gamma_2^{-1} = {}^{\mathrm{T}}\lambda'$. Then

$$m(C^{-1}D)[\lambda] = m{}^{\mathrm{T}}\lambda'\begin{pmatrix}\dfrac{u}{\alpha_1\alpha_2\beta_3} & \dfrac{x_0}{\alpha_1\beta_3\delta_2} \\ \dfrac{x_0}{\alpha_1\beta_3\delta_2} & \dfrac{w}{\delta\alpha_3/\beta_3}\end{pmatrix}\lambda' = m\left[\frac{\lambda_1'^2 u}{\alpha_1\alpha_2\beta_3} + \frac{2\lambda_1'\lambda_2' x_0}{\alpha_1\beta_3\delta_2} + \frac{\lambda_2'^2 w}{\delta\alpha_3/\beta_3}\right]. \tag{7.5}$$

Also,

$$-{}^{\mathrm{T}}vC^{-1}D\lambda = \frac{\lambda_1' rlu}{\alpha_1\alpha_2\beta_3} + \frac{x_0(\lambda_2' rl - \lambda_1' rv\alpha_3/\beta_3)}{\alpha_1\beta_3\delta_2} - \frac{w\lambda_2' rv}{\delta} \tag{7.6}$$

and

$$N^{-1}\mathrm{tr}(U_1(n_1)C^{-1}D) = \frac{n_1}{N}\left[\frac{l^2 u}{\alpha_1\alpha_2\beta_3} - \frac{2x_0 lv\alpha_3/\beta_3}{\alpha_1\beta_3\delta_2} + \frac{wv^2\alpha_3/\beta_3}{\delta}\right]. \tag{7.7}$$

Collecting (7.5), (7.6) and (7.7) we have

$$m(C^{-1}D)[\lambda] - {}^{\mathrm{T}}vC^{-1}D\lambda + N^{-1}\mathrm{tr}(U_1(n_1)C^{-1}D) =$$

$$uQ_u(\lambda_1')(\alpha_1\alpha_2\beta_3)^{-1} + x_0Q_x(\lambda_1', \lambda_2')(\alpha_1\beta_3\delta_2)^{-1} + wQ_w(\lambda_2')(\delta\alpha_3/\beta_3)^{-1},$$

where

$$Q_u(\lambda_1') = m\lambda_1'^2 + rl\lambda_1' + n_1N^{-1}l^2,$$

$$Q_x(\lambda_1', \lambda_2') = 2m\lambda_1'\lambda_2' + rl\lambda_2' - rv(\alpha_3/\beta_3)\lambda_1' - 2lv(\alpha_3/\beta_3)n_1N^{-1}$$

and

$$Q_w(\lambda_2') = m\lambda_2'^2 - rv(\alpha_3/\beta_3)\lambda_2' + v^2(\alpha_3/\beta_3)^2 n_1 N^{-1} .$$

Thus substituting into (7.4) and summing over u, x_0 and w we have

$$S(C, U_1(n_1), v) = (\alpha_1^2\delta_1)(\alpha_2\delta_2^2)(\alpha_3\beta_3\delta_3)\mathbf{N}(\alpha, \beta, \delta; D) , \tag{7.8}$$

where

$$\mathbf{N}(\alpha, \beta, \delta; D) = \sum_{\substack{\lambda \in \mathbf{Z}^2/{}^{\mathrm{T}}C\mathbf{Z}^2 \\ Q_u(\lambda_1') \equiv 0 \bmod \alpha_1\alpha_2\beta_3 \\ Q_x(\lambda_1', \lambda_2') \equiv 0 \bmod \alpha_1\beta_3\delta_2 \\ Q_w(\lambda_2') \equiv 0 \bmod \delta\alpha_3/\beta_3}} 1 .$$

The classes of λ in $\mathbf{Z}^2/{}^{\mathrm{T}}C\mathbf{Z}^2$ correspond to classes of λ_1' modulo $\alpha_1\alpha_2\beta_3$ and λ_2' modulo $\delta\alpha_3/\beta_3$. If we then write $\lambda_1' = \lambda_3'\alpha_1\alpha_2 + \lambda_3\beta_3$ and $\lambda_2' = \lambda_4'\delta_1\delta_2 + \lambda_4\delta_3\alpha_3/\beta_3$, taking λ_3 modulo $\alpha_1\alpha_2$, λ_3' modulo β_3, λ_4 modulo $\delta_1\delta_2$ and λ_4' modulo $\delta_3\alpha_3/\beta_3$, then using the congruence relations on l and v given in Lemma 7.2, we have the factorization

$\mathbf{N}(\alpha, \beta, \delta; D)$

$$= \left[\sum_{\substack{\lambda_3 \bmod \alpha_1\alpha_2,\ \lambda_4 \bmod \delta_1\delta_2, \\ m\lambda_3^2 \equiv 0 \bmod \alpha_1\alpha_2 \\ 2m\lambda_3\lambda_4 - r\lambda_3 \equiv 0 \bmod \alpha_1\delta_2 \\ m\lambda_4^2 - r\lambda_4 + n_1N^{-1} \equiv 0 \bmod \delta_1\delta_2}} 1\right]\left[\sum_{\substack{\lambda_3' \bmod \beta_3,\ \lambda_4' \bmod \delta_3\alpha_3/\beta_3 \\ m\lambda_3'^2 + r\lambda_3' + n_1N^{-1} \equiv 0 \bmod \beta_3 \\ 2m\lambda_3'\lambda_4' + r\lambda_4' - r(\alpha_3/\beta_3)\lambda_3' - 2(\alpha_3/\beta_3)n_1N^{-1} \equiv 0 \bmod \beta_3 \\ m\lambda_4'^2 - r(\alpha_3/\beta_3)\lambda_4' + (\alpha_3/\beta_3)^2 n_1 N^{-1} \equiv 0 \bmod \delta_3\alpha_3/\beta_3}} 1\right].$$

These two pieces now easily factor into primes by the Chinese remainder theorem, and substituting into (7.8) we obtain

$$S(C, U_1(n_1), v) = \prod_{p \mid \det C} S\left(\begin{pmatrix} p^a & p^b \\ & p^d \end{pmatrix}, U_1(n_1), v\right)$$

where

$$S\left(\begin{pmatrix} p^a & p^b \\ & p^d \end{pmatrix}, U_1(n_1), v\right) = \begin{cases} p^{2a+d}\mathbf{N}_1(p^a, p^b, p^d; D) & \text{if } p \in \Sigma_1, \\ p^{a+2d}\mathbf{N}_2(p^a, p^b, p^d; D) & \text{if } p \in \Sigma_2, \\ p^{a+b+d}\mathbf{N}_3(p^a, p^b, p^d; D) & \text{if } p \in \Sigma_3, \end{cases} \tag{7.9}$$

and for $i = 1, 2$

$$\mathbf{N}_i(p^a, p^b, p^d; D) = \sum_{\substack{\lambda_1 \bmod p^a,\ \lambda_2 \bmod p^d \\ m\lambda_1^2 \equiv 0 \bmod p^a \\ 2m\lambda_1\lambda_2 - r\lambda_1 \equiv 0 \bmod p^{\min(a,b)} \\ m\lambda_2^2 - r\lambda_2 + n_1N^{-1} \equiv 0 \bmod p^d}} 1, \tag{7.10}$$

while for $i = 3$,

$$\mathbf{N}_3(p^a, p^b, p^d; D) = \sum_{\substack{\lambda_1 \bmod p^b,\ \lambda_2 \bmod p^{a+d-b} \\ m\lambda_1^2 + r\lambda_1 + n_1 N^{-1} \equiv 0 \bmod p^b \\ 2m\lambda_1\lambda_2 + r\lambda_2 - rp^{a-b}\lambda_1 - 2p^{a-b}n_1 N^{-1} \equiv 0 \bmod p^b \\ m\lambda_2^2 - rp^{a-b}\lambda_2 + p^{2(a-b)}n_1 N^{-1} \equiv 0 \bmod p^{a+d-b}}} 1. \tag{7.11}$$

Notice that by the above $S(C, U_1(n_1), v)$ is almost independent of the choice of $\beta \bmod \delta$. Specifically we have, for $\gcd(u, p) = 1$,

$$S\left(\begin{pmatrix} p^a & up^b \\ & p^d \end{pmatrix}, U_1(n_1), v\right) = \begin{cases} S\left(\begin{pmatrix} p^a & p^d \\ & p^d \end{pmatrix}, U_1(n_1), v\right) & \text{if } a \leqq b, \\ S\left(\begin{pmatrix} p^a & p^b \\ & p^d \end{pmatrix}, U_1(n_1), v\right) & \text{if } a > b\,. \end{cases} \tag{7.12}$$

It follows from this minimal dependence on β that if $C = C'C''$ with $\gcd(\det C', \det C'') = 1$, then

$$S(C'C'', U_1(n_1), v) = S(C', U_1(n_1), v)S(C'', U_1(n_1), v) = S(C''C', U_1(n_1), v)\,.$$

Combining this multiplicativity with the Möbius inversion formula (7.3) we obtain

$$S_1\left(\begin{pmatrix} \alpha & \beta \\ & \delta \end{pmatrix}, U_1(n_1), v\right) = \prod_{p|\delta} \left[\sum_{H \left| \begin{pmatrix} p^a & p^b \\ & p^d \end{pmatrix} \right.} \mu(H) S\left(H^{-1}\begin{pmatrix} p^a & p^b \\ & p^d \end{pmatrix}, U_1(n_1), v\right) \right], \tag{7.13}$$

where the product is over $p|\delta$ because of the condition in (7.1) that $\alpha|\delta$.

The possible candidates for H are, for a given p,

$$\begin{pmatrix} p & u' \\ & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & \\ & p \end{pmatrix}, \quad \begin{pmatrix} p & \\ & p \end{pmatrix}, \quad \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$$

where u' varies modulo p. Then by (7.12) and (7.13) and the definition of $\mu(H)$ from Section 4, we may write

$$S_1\left(\begin{pmatrix} \alpha & \beta \\ & \delta \end{pmatrix}, U_1(n_1), v\right) = \prod_{p|\delta} S_p\left(\begin{pmatrix} p^a & p^b \\ & p^d \end{pmatrix}, D\right), \tag{7.14}$$

where if $d \geqq 1$ and if $a = 0$ or $b = 0$ then

$$S_p\left(\begin{pmatrix} p^a & p^b \\ & p^d \end{pmatrix}, D\right) = S\left(\begin{pmatrix} p^a & p^b \\ & p^d \end{pmatrix}, U_1(n_1), v\right) - pS\left(\begin{pmatrix} p^a & p^b \\ & p^{d-1} \end{pmatrix}, U_1(n_1), v\right), \tag{7.15}$$

while if $d \geqq 1$ and $a, b \geqq 1$

$$\begin{aligned} S_p\left(\begin{pmatrix} p^a & p^b \\ & p^d \end{pmatrix}, D\right) = {} & S\left(\begin{pmatrix} p^a & p^b \\ & p^d \end{pmatrix}, U_1(n_1), v\right) - p^2 S\left(\begin{pmatrix} p^{a-1} & p^{b-1} \\ & p^d \end{pmatrix}, U_1(n_1), v\right) \\ & - pS\left(\begin{pmatrix} p^a & p^b \\ & p^{d-1} \end{pmatrix}, U_1(n_1), v\right) + p^3 S\left(\begin{pmatrix} p^{a-1} & p^{b-1} \\ & p^{d-1} \end{pmatrix}, U_1(n_1), v\right). \end{aligned} \tag{7.16}$$

Before turning back to the evaluation of the inner sums $S\left(\begin{pmatrix} p^a & p^b \\ & p^d \end{pmatrix}, U_1(n_1), v\right)$, we note that the Dirichlet series $\mathscr{L}(s, D)$ is now clearly seen to be an Euler product. In addition, if we write $\beta = u'p^b$ with $\gcd(u', p) = 1$, then the independence of $S_1\left(\begin{pmatrix} p^a & u'p^b \\ & p^d \end{pmatrix}, U_1(n_1), v\right)$ of u', and of b if $a \leqq b$ means that we may consider, in the p-part of (7.1), the inner sum

$$\sum_{\substack{\beta \bmod p^d \\ \beta = up^a \\ \beta \equiv 0 \bmod N}} e\left(\frac{N^{-1}\beta}{p^a}\right) = p^{d-a}, \tag{7.17}$$

corresponding to all $b \geqq a$ and a fixed $b < a$

$$\sum_{\substack{\beta \bmod p^d \\ \beta \equiv 0 \bmod N \\ \beta = u'p^b \\ \gcd(u', p) = 1}} e\left(\frac{N^{-1}\beta}{p^a}\right) = \begin{cases} -p^{d-a} & \text{if } a = b+1, \\ 0 & \text{if } a > b+1. \end{cases} \tag{7.18}$$

Recall that as $\alpha | \delta$ we have $a \leqq \delta$ for every p. Combining (7.17) and (7.18) and (7.14) and the independence described above, the series in (7.1) factors as

$$\mathscr{L}(s, D) = \prod_{p \nmid N} \mathscr{L}_p(s, D) \tag{7.19}$$

where

$$\mathscr{L}_p(s, D) = 1 + \sum_{d=1}^{\infty} \sum_{0 \leqq a \leqq d} p^{d-a}\left[S_p\left(\begin{pmatrix} p^a & p^d \\ & p^d \end{pmatrix}, D\right) - S_p\left(\begin{pmatrix} p^a & p^{a-1} \\ & p^d \end{pmatrix}, D\right)\right] \times$$
$$p^{-(a+d)s-(d-a)k/2} a(p^{d-a}). \tag{7.20}$$

The second S_p term is only present in the above if $a \geqq 1$.

It remains now to complete the evaluation of the $S\left(\begin{pmatrix} p^a & p^b \\ & p^d \end{pmatrix}, U_1(n_1), v\right)$ sums, and hence of the $S_p\left(\begin{pmatrix} p^a & p^b \\ & p^d \end{pmatrix}, D\right)$ sums. Note that we are reduced to the two cases $b = d$ and $b = a - 1$. Recall that we are assuming that $4|N$ and $N|m$. We will assume henceforth that $p \nmid m$, so in particular, $p \neq 2$. Write $D = r^2 - 4mn_1N^{-1} = D'p^{2k_1}$ with $p^2 \nmid D'$. Then for some r_0, n_0 we may write $D' = r_0^2 - 4mn_0N^{-1}$ and as the Dirichlet series depends only on D we may, if necessary, change r and n_1 so that $r = r_0p^{k_1}$, $n_1 = n_0p^{2k_1}$. Because of (7.9) we may concentrate on the $\mathbf{N}_i(p^a, p^b, p^d; D)$, which we evaluate in the following Lemma. Notice that in evaluating $\mathbf{N}_3$ we may restrict ourselves to the cases $a = b + 1$ with $a \leqq d$, as if $a > d$, $a = b + 1$, then the only relevant case in our future computations is $a = d + 1$, $b = d$, which can be included in the discussion of $\mathbf{N}_2$ after multiplying the corresponding $\begin{pmatrix} p^a & p^b \\ & p^d \end{pmatrix}$ on the left by an element of $\Gamma^0(N)$. We will use the notation, for $n \in \mathbf{Z}$,

$$\varepsilon(n) = \begin{cases} 1 & \text{if } n \text{ is odd}, \\ 0 & \text{if } n \text{ is even}. \end{cases}$$

Lemma 7.3. *For $p \nmid m$ let $\mathbf{N}_i(p^a, p^b, p^d; D)$ be as defined by* (7.10) *and* (7.11), *and $D = D'p^{2k_1}$ with $p^2 \nmid D'$. Then if $i = 1$ or 2 and $a \leqq d + 1$,*

$\mathbf{N}_i(p^a, p^b, p^d; D'p^{2k_1}) =$

$$\begin{cases} p^{(d-\varepsilon(d)+a-\varepsilon(a))/2} & \text{if } d \leqq 2k_1, \\ 2p^{k_1+\min((a-\varepsilon(a))/2,k_1)} & \text{if } \chi_{D'}(p) = 1,\ d \geqq 2k_1 + 1 \text{ and } i = 1; \\ 2p^{2k_1+1} & \text{if } \chi_{D'}(p) = 1,\ d \geqq 2k_1 + 1,\ i = 2; \\ p^{k_1+(a-\varepsilon(a))/2} & \text{if } \chi_{D'}(p) = 0,\ d = 2k_1 + 1,\ i = 1; \\ p^{2k_1+1} & \text{if } \chi_{D'}(p) = 0,\ d = 2k_1 + 1,\ i = 2; \\ 0 & \text{otherwise}, \end{cases}$$

while if $i = 3$ and $b \leqq d - 1$, $d \geqq 1$

$\mathbf{N}_3(p^{b+1}, p^b, p^d; D'p^{2k_1}) =$

$$\begin{cases} p^{(d+\varepsilon(d)+b-\varepsilon(b))/2} & \text{if } d \leqq 2k_1 + 1,\ b \leqq 2k_1; \\ 2p^{k_1+1+(b-\varepsilon(b))/2} & \text{if } \chi_{D'}(p) = 1,\ d \geqq 2k_1 + 2,\ b \leqq 2k_1; \\ p^{k_1+1+(b-\varepsilon(b))/2} & \text{if } \chi_{D'}(p) = 0,\ d = 2k_1 + 2,\ b \leqq 2k_1; \\ 4p^{2k_1+1} & \text{if } \chi_{D'}(p) = 1,\ d \geqq 2k_1 + 2,\ b = 2k_1 + 1; \\ 2p^{2k_1+1} & \text{if } \chi_{D'}(p) = 1,\ d \geqq 2k_1 + 2,\ b \geqq 2k_1 + 2; \\ p^{2k_1+1} & \text{if } \chi_{D'}(p) = 0,\ d = 2k_1 + 2,\ b = 2k_1 + 1; \\ 0 & \text{otherwise}. \end{cases}$$

Proof. The proof of the Lemma is intricate but not difficult. We will give details for the case $i = 1$ and for part of the case $i = 3$. Recall from (7.10) that when $i = 1$ the relations which must be simultaneously satisfied are

$$m\lambda_1^2 \equiv 0 \bmod p^a, \tag{7.21}$$

$$2m\lambda_1\lambda_2 - r_0 p^{k_1}\lambda_1 \equiv 0 \bmod p^a, \tag{7.22}$$

$$m\lambda_2^2 - r_0 p^{k_1}\lambda_2 + n_0 N^{-1} p^{2k_1} \equiv 0 \bmod p^d, \tag{7.23}$$

for λ_1 modulo p^a and λ_2 modulo p^d. If $d \leqq 2k_1$, then (7.23) reduces to $\lambda_2(m\lambda_2 - r_0 p^{k_1}) \equiv 0 \bmod p^d$, whose solutions are of the form $\lambda_2 = \lambda_2' p^{(d+\varepsilon(d))/2}$, for all λ_2' modulo $p^{(d-\varepsilon(d))/2}$. Then (7.21) has solutions of the form $\lambda_1 = \lambda_1' p^{(a+\varepsilon(a))/2}$ for λ' modulo $p^{(a-\varepsilon(a))/2}$ and we observe that for such λ_1, λ_2 equation (7.22) is always satisfied. Thus the total number of solutions to (7.21–23) in the case $i = 1, d \leqq 2k_1$ is $p^{(d-\varepsilon(d))/2} \cdot p^{(a-\varepsilon(a))/2}$ as indicated above. If $d \geqq 2k_1 + 1$ write $d = 2k_1 + n$ with $n \geqq 1$. For (7.23) to hold we must have $\lambda_2 = \lambda_2' p^{k_1}$ for λ_2' modulo p^{k_1+n}, and (7.23) then reduces to

$$m\lambda_2'^2 - r_0\lambda_2' + n_0 N^{-1} \equiv 0 \bmod p^n. \tag{7.24}$$

The number of solutions to (7.24) modulo p^n is then 2 if $\chi_{D'}(p) = 1$, 1 if $\chi_{D'}(p) = 0$ and $n = 1$, and 0 otherwise, and as λ_2' is taken modulo p^{k_1+n} there is a multiplicity of p^{k_1} and we have

$$\text{The number of solutions to (7.24)} = \begin{cases} 2p^{k_1} & \text{if } \chi_{D'}(p) = 1, \\ p^{k_1} & \text{if } \chi_{D'}(p) = 0,\ n = 1\ , \\ 0 & \text{otherwise}\ . \end{cases} \quad (7.25)$$

The case $i = 1$, $d \geqq 2k_1 + 1$ is now complete if $a = 0$, so suppose henceforth that $a \geqq 1$. As in the case $d \leqq 2k_1$ the solutions to (7.21) are given by $\lambda_1 = \lambda_1' p^{(a+\varepsilon(a))/2}$, λ_1' modulo $p^{(a-\varepsilon(a))/2}$, and hence (7.22) becomes

$$\lambda_1' p^{k_1 + (a+\varepsilon(a))/2}(2m\lambda_2' - r_0) \equiv 0 \bmod p^a\ . \quad (7.26)$$

Suppose now that $\chi_{D'}(p) = 1$. Then $p \nmid 2m\lambda_2' - r_0$ and in order that (7.26) be satisfied we require either that $k_1 + (a + \varepsilon(a))/2 \geqq a$, in which case the number of possible λ_1' is $p^{(a-\varepsilon(a))/2}$, or else $k_1 < (a - \varepsilon(a))/2$, in which case we must have $\lambda_1' = \lambda_1'' p^{(a-\varepsilon(a))/2 - k_1}$ and there are p^{k_1} possible λ_1'. Multiplying the above by (7.25) gives the case $\chi_{D'}(p) = 1$, $d \geqq 2k_1 + 1$ when $i = 1$. If $\chi_{D'}(p) = 0$ and $n = 1$ then $d = 2k_1 + 1$ is odd. We have $a \leqq d$ so if a is even then $a \leqq 2k_1$, while if a is odd $a \leqq 2k_1 + 1$, so $k_1 \geqq (a - \varepsilon(a))/2$. Then

$$k_1 + (a + \varepsilon(a))/2 \geqq (a - \varepsilon(a))/2 + (a + \varepsilon(a))/2 = a\ .$$

Thus (7.26) will be always satisfied when $\chi_{D'}(p) = 0$ and multiplying the $p^{(a-\varepsilon(a))/2}$ solutions of (7.21) by p^{k_1} from (7.25) gives the number of solutions to (7.21–23) when $\chi_{D'}(p) = 0$. This completes the proof of the Lemma in the first case $i = 1$. The case $i = 2$ is virtually identical, and we omit the details.

When $i = 3$ we have from (7.11) the relations

$$m\lambda_1^2 + r_0 p^{k_1}\lambda_1 + n_0 N^{-1} p^{2k_1} \equiv 0 \bmod p^b\ , \quad (7.27)$$

$$2m\lambda_1\lambda_2 + r_0 p^{k_1}\lambda_2 - r_0 p^{k_1+1}\lambda_1 - 2p^{2k_1+1} n_0 N^{-1} \equiv 0 \bmod p^b\ , \quad (7.28)$$

$$m\lambda_2^2 - r_0 p^{k_1+1}\lambda_2 + p^{2k_1+2} n_0 N^{-1} \equiv 0 \bmod p^{d+1}\ , \quad (7.29)$$

with λ_1 modulo p^b, λ_2 modulo p^{d+1}. Equation (7.29) forces $\lambda_2 = \lambda_2' p$ with λ_2' modulo p^d and so (7.29) becomes

$$m\lambda_2'^2 - r_0 p^{k_1}\lambda_2' + p^{2k_1} n_0 N^{-1} \equiv 0 \bmod p^{d-1}\ , \quad (7.30)$$

while (7.28) reduces to

$$2m\lambda_1\lambda_2' + r_0 p^{k_1}\lambda_2' - r_0 p^{k_1}\lambda_1 - 2p^{2k_1} n_0 N^{-1} \equiv 0 \bmod p^{b-1} \quad (7.31)$$

if $b \geqq 1$, and is trivially satisfied if $b = 0$. If $b \leqq 2k_1$, then the analysis of the first and third equations (7.27) and (7.30) proceeds exactly as before, and the middle equation (7.31) will be satisfied whenever the first and third are. The only new element enters if $b = 2k_1 + n'$ and $d - 1 = 2k_1 + n$ with $n \geqq n' \geqq 1$. Then we must have $\lambda_2' = \lambda_2'' p^{k_1}$, $\lambda_1 = \lambda_1' p^{k_1}$ with λ_2'' modulo p^{k_1+n+1}, λ_1' modulo $p^{k_1+n'}$, and the three equations become

$$\begin{aligned} m\lambda_1'^2 + r_0\lambda_1' + n_0 N^{-1} &\equiv 0 \bmod p^{n'}\ , \\ 2m\lambda_1'\lambda_2'' + r_0\lambda_2'' - r_0\lambda_1' - 2n_0 N^{-1} &\equiv 0 \bmod p^{n'-1}\ , \\ m\lambda_2''^2 - r_0\lambda_2'' + n_0 N^{-1} &\equiv 0 \bmod p^n\ . \end{aligned} \quad (7.32)$$

If $\chi_{D'}(p) = 0$ then $n = n' = 1$ is forced so (7.32) is trivially satisfied and the number of solutions is $p^{k_1+1} \cdot p^{k_1} = p^{2k_1+1}$. If $\chi_{D'}(p) = 1$ and $n' = 1$ then again (7.32) is trivially satisfied and there are $(2p^{k_1+1})(2p^{k_1}) = 4p^{2k_1+1}$ solutions. If $\chi_{D'}(p) = 1$ and $n' \geqq 2$ then λ_2'' and λ_1' have the form

$$\lambda_1' \equiv \frac{1}{2m}(-r_0 + \varepsilon_1 t) \bmod p ,$$

$$\lambda_2'' \equiv \frac{1}{2m}(r_0 + \varepsilon_2 t) \bmod p ,$$

where t is a solution to the congruence $t^2 \equiv D' \bmod p$, and $\varepsilon_1, \varepsilon_2 = \pm 1$. Substituting into (7.32) we see that the equation will be satisfied if and only if $\varepsilon_1\varepsilon_2 = -1$. Thus there are two possible pairs of solutions modulo p, and counting multiplicities the total number of solutions is $2p^{2k_1+1}$. This completes the proof of Lemma 7.3. □

Lemma 7.4. *If* $d \geqq 1$ *then*

$$S_p\left(\begin{pmatrix} 1 & p^d \\ & p^d \end{pmatrix}, D'p^{2k_1}\right) = p^d \times \begin{cases} p^{d/2}(1-p^{-1}) & \text{if } 2 \leqq d \leqq 2k_1,\ d \text{ even;} \\ \chi_{D'}(p)p^{k_1} & \text{if } d = 2k_1+1; \\ -p^{k_1} & \text{if } d = 2k_1+2,\ \chi_{D'}(p) = 0; \\ 0 & \text{otherwise .} \end{cases}$$

If $1 \leqq a \leqq d-1$ *then*

$$S_p\left(\begin{pmatrix} p^a & p^d \\ & p^d \end{pmatrix}, D'p^{2k_1}\right) =$$

$$p^{2a+d} \times \begin{cases} p^{(a+d)/2}(1-p^{-1})^2 & \text{if } d \leqq 2k_1,\ a \text{ even}, d \text{ even;} \\ \chi_{D'}(p)p^{k_1+a/2}(1-p^{-1}) & \text{if } d = 2k_1+1,\ a \text{ even;} \\ -p^{k_1+a/2}(1-p^{-1}) & \text{if } d = 2k_1+2,\ a \text{ even},\ \chi_{D'}(p) = 0; \\ 0 & \text{otherwise .} \end{cases}$$

$$S_p\left(\begin{pmatrix} p^d & p^d \\ & p^d \end{pmatrix}, D'p^{2k_1}\right) = p^{3d} \times \begin{cases} p^{d-1}(1-p^{-1}) & \text{if } d \leqq 2k_1-1,\ d \text{ odd;} \\ p^d(1-p^{-1}) & \text{if } d \leqq 2k_1,\ d \text{ even;} \\ p^{2k_1}(1-p^{-1}) & \text{if } d = 2k_1+1; \\ 0 & \text{otherwise.} \end{cases}$$

$$S_p\left(\begin{pmatrix} p & 1 \\ & p^d \end{pmatrix}, D'p^{2k_1}\right) = p^{d+1} \times \begin{cases} p^{(d+1)/2}(1-p^{-1}) & \text{if } d \leqq 2k_1+1,\ d \text{ odd;} \\ \chi_{D'}(p)p^{k_1+1} & \text{if } d = 2k_1+2; \\ -p^{k_1+1} & \text{if } d = 2k_1+3,\ \chi_{D'}(p) = 0; \\ 0 & \text{otherwise .} \end{cases}$$

If $1 \leqq b \leqq d-2$ *then* $S_p\left(\begin{pmatrix} p^{b+1} & p^b \\ & p^d \end{pmatrix}, D'p^{2k_1}\right) = p^{2b+d+1}$

$$\times \begin{cases} p^{(d+b+1)/2}(1-p^{-1})^2 & \text{if } b \text{ is even, } d \text{ odd, } d \leqq 2k_1+1; \\ \chi_{D'}(p)p^{k_1+1+b/2}(1-p^{-1}) & \text{if } b \text{ is even, } d = 2k_1+2; \\ -p^{k_1+1+b/2}(1-p^{-1}) & \text{if } b \text{ is even, } d = 2k_1+3,\ b \leqq 2k_1,\ \chi_{D'}(p) = 0; \\ 0 & \text{otherwise .} \end{cases}$$

If $d \geqq 2$ *then*

$$S_p\left(\begin{pmatrix} p^d & p^{d-1} \\ & p^d \end{pmatrix}, D'p^{2k_1}\right) = p^{3d-1} \times \begin{cases} p^d(1-p^{-1})^2 & \text{if } d \leqq 2k_1+1,\ d \text{ odd;} \\ p^{2k_1+1} & \text{if } \chi_{D'}(p) = \pm 1,\ d = 2k_1+2; \\ p^{2k_1+1} & \text{if } \chi_{D'}(p) = 0,\ d = 2k_1+3; \\ 0 & \text{otherwise}\ . \end{cases}$$

Proof. We may use Lemma 7.3, together with (7.9), (7.15) and (7.16) to compute the values of $S_p\left(\begin{pmatrix} p^a & p^b \\ & p^d \end{pmatrix}, D\right)$. This is a long but routine calculation and so we omit the details. □

We will now use this Lemma to compute the p-factor $\mathscr{L}_p(s, D)$ of (7.20) in the case $k_1 = 0$. We are assuming now that $D = D_0 D_1^2$ where D_0 is a fundamental discriminant, that $p \nmid D_1$, and that $p \nmid m$. Note that if D' is as in Lemma 7.3, then $\chi_{D'}(p) = \chi_{D_0}(p)$.

First suppose that $\chi_{D_0}(p) \neq 0$. Then by Lemma 7.4 we see that the following is a complete list of all nonzero $S_p\left(\begin{pmatrix} p^a & p^d \\ & p^d \end{pmatrix}, D\right)$ and $S_p\left(\begin{pmatrix} p^{b+1} & p^b \\ & p^d \end{pmatrix}, D\right)$ with $d \geqq 1$:

$$S_p\left(\begin{pmatrix} 1 & p \\ & p \end{pmatrix}, D\right) = \chi_{D_0}(p)p, \qquad S_p\left(\begin{pmatrix} p & p \\ & p \end{pmatrix}, D\right) = p^3 - p^2\,,$$

$$S_p\left(\begin{pmatrix} p & 1 \\ & p \end{pmatrix}, D\right) = p^3 - p^2, \qquad S_p\left(\begin{pmatrix} p & 1 \\ & p^2 \end{pmatrix}, D\right) = \chi_{D_0}(p)p^4\,,$$

$$S_p\left(\begin{pmatrix} p^2 & p \\ & p^2 \end{pmatrix}, D\right) = p^6\,.$$

Therefore, by (7.20), for $\chi_{D_0}(p) \neq 0$, i.e. for $p \nmid D_0$, we have

$$\begin{aligned} \mathscr{L}_p(s, D) &= 1 + \chi_{D_0}(p)p^{2-k/2-s}a(p) - \chi_{D_0}(p)p^{5-k/2-3s}a(p) - p^{6-4s} \\ &= (1 + \chi_{D_0}(p)\sigma_p p^{2-k/2-s})(1 + \chi_{D_0}(p)\sigma'_p p^{2-k/2-s})(1 - p^{3-2s})\,. \end{aligned} \tag{7.33}$$

Similarly if $\chi_{D_0}(p) = 0$ we have

$$S_p\left(\begin{pmatrix} p & p \\ & p \end{pmatrix}, D\right) = p^3 - p^2, \qquad S_p\left(\begin{pmatrix} p & 1 \\ & p \end{pmatrix}, D\right) = p^3 - p^2\,,$$

$$S_p\left(\begin{pmatrix} 1 & p^2 \\ & p^2 \end{pmatrix}, D\right) = -p^2, \qquad S_p\left(\begin{pmatrix} p & 1 \\ & p^3 \end{pmatrix}, D\right) = -p^5\,,$$

$$S_p\left(\begin{pmatrix} p^3 & p^2 \\ & p^3 \end{pmatrix}, D\right) = p^9\,,$$

and thus when $p | D_0$ we have

$$\begin{aligned} \mathscr{L}_p(s, D) &= 1 - p^{4-k-2s}a(p^2) + p^{7-k-4s}a(p^2) - p^{9-6s} \\ &= (1 - \sigma_p^2 p^{4-k-2s})(1 - \sigma_p'^2 p^{4-k-2s})(1 - p^{3-2s})\,. \end{aligned} \tag{7.34}$$

Equations (7.33–34), when substituted into (7.19) give the description of $\mathscr{L}(s, D)$ and $\mathscr{L}(s, D_0)$ mentioned in Proposition 7.1. The Dirichlet polynomials $d(s, D_1)$ of Proposition 7.1 are given by

$$d(s, D_1) = \prod_{p|D_1} \mathscr{L}_p(s, D) \tag{7.35}$$

where by Lemma 7.4 each $\mathscr{L}_p(s, D)$ is a finite sum given by (7.20).

The final case of interest to us is when $D = 0$. We may read this off from Lemma 7.4 by taking $k_1 = \infty$. Doing this we obtain, for $d \geqq 1$.

$$S_p\left(\begin{pmatrix} 1 & p^d \\ & p^d \end{pmatrix}, 0\right) = p^{3d/2}(1 - p^{-1}) \qquad \text{if } d \geqq 2,\ d \text{ even},$$

$$S_p\left(\begin{pmatrix} p^a & p^d \\ & p^d \end{pmatrix}, 0\right) = p^{(3d+5a)/2}(1 - p^{-1})^2 \qquad \text{if } d \geqq 4,\ 2 \leqq a \leqq d-2,\ a, d \text{ even},$$

$$S_p\left(\begin{pmatrix} p^d & p^d \\ & p^d \end{pmatrix}, 0\right) = \begin{cases} p^{4d-1}(1 - p^{-1}) & \text{if } d \text{ is odd}, \\ p^{4d}(1 - p^{-1}) & \text{if } d \text{ is even}, \end{cases}$$

$$S_p\left(\begin{pmatrix} p & 1 \\ & p^d \end{pmatrix}, 0\right) = p^{(3d+3)/2}(1 - p^{-1}) \qquad \text{if } d \text{ is odd},$$

$$S_p\left(\begin{pmatrix} p^{b+1} & p^b \\ & p^d \end{pmatrix}, 0\right) = p^{(3d+5b+3)/2}(1 - p^{-1})^2 \qquad \text{if } b \text{ is even, } d \text{ odd, } 2 \leqq b \leqq d-1.$$

Substituting into (7.20) we then have

$$\begin{aligned}
\mathscr{L}_p(s, 0) = 1 &+ \sum_{\substack{d \geqq 2 \\ d \text{ even}}} p^{(5d-dk)/2-ds} a(p^d)(1 - p^{-1}) + \sum_{\substack{d \geqq 2 \\ d \text{ even}}} p^{4d-2ds}(1 - p^{-1}) \\
&+ \sum_{\substack{d \geqq 3 \\ d \text{ odd}}} p^{4d-2-2ds}(1 - p^{-1}) \\
&- \sum_{\substack{d \geqq 3 \\ d \text{ odd}}} p^{(5d+1-(d-1)k)/2-(d+1)s} a(p^{d-1})(1 - p^{-1}) \\
&- \sum_{\substack{d \geqq 5 \\ 3 \leqq a \leqq d-2 \\ a, d \text{ odd}}} p^{(5d+3a-(d-a)k)/2-1-(d+a)s} a(p^{d-a})(1 - p^{-1})^2 \\
&+ \sum_{\substack{d \geqq 4 \\ 2 \leqq a \leqq d-2 \\ a, d \text{ even}}} p^{(5d+3a-(d-a)k)/2-(d+a)s} a(p^{d-a})(1 - p^{-1})^2.
\end{aligned}$$

A long but routine calculation now shows that

$$\mathscr{L}_p(s, 0) = \frac{(1 - \sigma_p^2 p^{4-k-2s})(1 - \sigma_p'^2 p^{4-k-2s})(1 - p^{3-2s})}{(1 - \sigma_p^2 p^{5-k-2s})(1 - \sigma_p'^2 p^{5-k-2s})(1 - p^{4-2s})}$$

which, substituted into (7.19) gives the evaluation of $\mathscr{L}(s, 0)$.

It only remains to demonstrate that if $s \geqq 2$ is real, then the series $\sum_{D_1=1}^{\infty} \mathscr{L}(s, D_0 D_1^2) D_1^{-2u}$ converges for $\mathrm{re}(u) > 3/4$. This can be done most elegantly by using the recursion relations mentioned in the Remark following the statement of Proposition 7.1 to express the latter summation as an Euler product equal to a ratio of L-functions. This gives the convergence back to $\mathrm{re}(u) > 1/2$, and the asserted nonvanishing property. However, since we will not prove these recursion relations in this paper, we will use instead the upper bounds provided by the evaluations in Lemma 7.4 and the bound

$$p^{-d(k-1)/2} a(p^d) \ll p^{d/4+\varepsilon} \tag{7.36}$$

on the Fourier coefficients, the implied constant depending only on ε. (The Ramanujan conjecture, which is known for holomorphic forms, would also imply convergence back to $\mathrm{re}(u) > 1/2$, but as this is not essential for our purpose we prefer to use the weaker bound which is known to be true also for Maass forms.)

Referring to Lemma 7.4 we see that

$$\left| S_p\left(\begin{pmatrix} p^a & p^b \\ & p^d \end{pmatrix}, D' p^{2k_1}\right)\right| \leqq \begin{cases} p^{(5a+3d)/2} & \text{if } d \leqq 2k_1 , \\ p^{(5a+3d)/2} & \text{if } d = 2k_1 + 1 , \\ p^{(5a+3d)/2-1} & \text{if } d = 2k_1 + 2 , \\ p^{(5a+3d)/2-2} & \text{if } d = 2k_1 + 3 , \end{cases}$$

and substituting these upper bounds into (7.20) we have, for $D = D_0 D_1'^2 p^{2k_1}$ with $\gcd(p, ND_1') = 1$,

$$\begin{aligned} |\mathscr{L}_p(s, D_0 D_1'^2 p^{2k_1})| \leqq\ & 2 \sum_{d \leqq 2k_1} \sum_{a \leqq d} p^{-(d-a)(k-1)/2} |a(p^{d-a})| \\ & + 2 \sum_{a \leqq 2k_1+1} p^{-((2k_1+1-a)(k-1)-1)/2} |a(p^{2k_1+1-a})| \\ & + 2 \sum_{a \leqq 2k_1+2} p^{-(2k_1+2-a)(k-1)/2-1} |a(p^{2k_1+2-a})| \\ & + 2 \sum_{a \leqq 2k_1+3} p^{-(2k_1+3-a)(k-1)/2-2} |a(p^{2k_1+3-a})| . \end{aligned}$$

The first of these sums is the largest. Using the coefficient bound (7.36) we obtain

$$|\mathscr{L}_p(s, D_0 D_1'^2 p^{2k_1})| \ll p^{k_1/2+\varepsilon} ,$$

with the implied constant depending only on ε. Thus

$$\prod_{\substack{p \mid D_1 \\ p \nmid N}} |\mathscr{L}_p(s, D_0 D_1^2)| \ll D_1^{1/2+\varepsilon} . \tag{7.37}$$

Now, for $p \nmid ND_1$

$$\mathscr{L}_p(s, D_0 D_1^2) = \mathscr{L}_p(s, D_0) = \frac{(1 - \sigma_p^2 p^{4-k-2s})(1 - \sigma_p'^2 p^{4-k-2s})(1 - p^{3-2s})}{(1 - \chi_{D_0}(p)\sigma_p p^{2-k/2-s})(1 - \chi_{D_0}(p)\sigma_p' p^{2-k/2-s})}$$

and so

$$\prod_{p \nmid ND_1} \mathscr{L}_p(s, D_0 D_1^2) = \frac{L_N(s + k/2 - 2, f, \chi_{D_0})}{L_N(2s + k - 4, f, \vee^2)} \prod_{\substack{p \mid D_1 \\ p \nmid N}} \frac{H_p(s + k/2 - 2, f, \chi_{D_0})}{H_p(2s + k - 4, f, \vee^2)} , \tag{7.38}$$

where the factors are finite Dirichlet polynomials equal to the inverses of the corresponding p-factors for $L_N(s + k/2 - 2, f, \chi_{D_0})$ and $L_N(2s + k - 4, f, \vee^2)$. We note that $L_N(2s + k - 4, f, \vee^2)$ is nonzero and analytic for real $s \geqq 2$ by Proposition 1.2. Moreover, the estimate $|\sigma'_p| < p^{k/2 - 1/4}$, which is a slight strengthening of (7.36) is well known and therefore we have

$$\prod_{\substack{p|D_1 \\ p\nmid N}} \left| \frac{H_p(s + k/2 - 2, f, \chi_{D_0})}{H_p(2s + k - 4, f, \vee^2)} \right| \ll D_1^{\varepsilon} . \tag{7.39}$$

Combining (7.37), (7.38) and (7.39) gives

$$|\mathscr{L}(s, D_0 D_1^2)| \ll \left| \frac{L_N(s + k/2 - 2, f, \chi_{D_0})}{L_N(2s + k - 4, f, \vee^2)} \right| D_1^{1/2+\varepsilon} , \tag{7.40}$$

with the implied constant depending only on ε. Finally, by (7.40), for real $s \geqq 2$

$$\sum_{D_1 = 1}^{\infty} |\mathscr{L}(s, D_0 D_1^2) D_1^{-2u}| \ll \left| \frac{L_N(s + k/2 - 2, f, \chi_{D_0})}{L_N(2s + k - 4, f, \vee^2)} \right| \sum_{D_1 = 1}^{\infty} D_1^{-(2u - 1/2 - \varepsilon)} , \tag{7.41}$$

which converges for $\mathrm{re}(u) \geqq 3/4 + \varepsilon/2$.

If $\mathscr{L}(2, D_2) = 0$, then $L_N(\frac{k}{2}, f, \chi_{D_0}) = 0$ and so $\mathscr{L}(2, D_0 D_1^2) = 0$ for all D_1. Differentiating and using the product rule for derivatives, we obtain an upper bound for $\sum_{D_1=1}^{\infty} |\mathscr{L}'(s, D_0 D_1^2) D_1^{-2u}|$ identical to (7.41) with $L_N(s + \frac{k}{2} - 2, f, \chi_{D_0})$ replaced by $L'_N(\frac{k}{2}, f, \chi_{D_0})$. This completes the proof of Proposition 7.1. □

8. The Novodvorsky transform

Novodvorsky [16] considered the problem of representing the Langlands L-function of degree four associated with an automorphic form on $GSp(4)$. Provided that the form is *generic* in the sense of having a Whittaker model, he found an integral with analytic continuation and functional equation representing the L-function. See also Bump [2], Section 3.3 for details concerning Novodvorsky's integral.

We will assume given a representation $\sigma\colon K \to GL(\mathbf{V})$, a vector $\mathbf{v} \in \mathbf{V}$ satisfying (1.12), a linear functional T on $\mathbf{V}$, and a fixed positive value y_2. *We will assume that the matrix coefficient $\phi(\kappa) = T(\mathbf{v}\sigma(\kappa)$ is divisible by ϕ_1 (defined by* (3.29)), *so that Proposition* 3.11 *is applicable.* (This hypothesis is certainly not necessary, but very convenient.)

Let $\Phi_j = E_s | J^j$ be the Jacobi modular Eisenstein series introduced in Section 1, and let $\mathscr{E}_j(g, \mu, s) = \mathscr{E}_j(g, \mu)$ be the functions associated with Φ_j by Proposition 2.1. $\mathscr{E}_1(g, \mu, s)$ may be regarded as an automorphic form on the metaplectic group—the double cover of $GSp(4)$. It is generic in the sense of having a Whittaker model. We may therefore apply Novodvorsky's construction to $\mathscr{E}_1(g, \mu, s)$. The resulting Dirichlet series (in a new variable u) has analytic continuation and functional equation. Naturally, it does not have an Euler product.

Let us first assume that $\mathrm{re}(s) > 3/2$, and that u has large real part. Let $v = {}^T(0, r)$, where r is an integer. We will define two Dirichlet series $Z^{\pm}(u, s)$ by

$$Z^{\pm}(u, s) = \sum_{\substack{D \equiv r^2 \bmod 4m\, N \\ \pm D > 0}} \mathscr{L}(s, D)\, |D|^{s - u - 5/2} ,$$

where $\mathscr{L}(s, D)$ was defined in Section 6, and evaluated in Section 7. With notation otherwise as in (3.39), (3.48) and (3.49), we will prove

Proposition 8.1. *If* $\mathrm{re}(s) > 3/2$, *the function*

$$(4m)^{-s+u+5/2} N^{-s+4+k/2} [Z^+(u, s)\, T\mathscr{F}^+(u, s, y_2) + Z^-(u, s)\, T\mathscr{F}^-(u, s, y_2)] \tag{8.1}$$

has meromorphic continuation to the region $\mathrm{re}(u) > 0$, $\mathrm{re}(u) > \mathrm{re}(s) - 5/2$ *and differs from*

$$\begin{aligned} &-(u - s + 5/2)^{-1} N^{-s+4+k/2} \mathscr{L}(s, 0)\, T\mathscr{M}(s, N^{-1}y_2; \mathrm{v}, \sigma) \\ &+ (u + s - 5/2)^{-1} N^{7-s-k/2} \hat{\mathscr{L}}(s, 0, r)\, y_2^{2s-5}\, T\hat{\mathscr{M}}(s, N^{-1}y_2; \mathrm{v}, \sigma) \\ &+ (u - s + 3/2)^{-1} N^{-s} y_2^{3-s+k/2}\, T\tau(s, N^{-1}y_2; \mathrm{v}, \sigma)\,. \end{aligned} \tag{8.2}$$

by a function of s and u which is holomorphic in a neighborhood of $(u, s) = (1/2, 2)$.

Proof. First we assume that re(u) is large. We will obtain an integral expression for the left hand side of (8.2) which is valid for all s and u in the region described in the Proposition. This expression will represent the left hand side of (8.2) as the three factors on the right, plus two other terms (denoted I_1 and I_5) which are very rapidly convergent series. At a key step in the calculation (equation (8.9) below) we will use the transformation property of the Eisenstein series. Thus the analytic continuation uses strongly the Selberg-Langlands theory of Eisensteins series, which guarantees that the Eisenstein series itself has analytic continuation, and poles only where the constant terms themselves have poles. The poles of the Eisenstein series are at the poles of $\mathscr{L}(s, 0)$ and $\hat{\mathscr{L}}(s, 0)$, which by Proposition 7.1 are at the zeros of the symmetric square L-function $L(2s + k - 4, f, \mathrm{v}^2)$. By Proposition 1.2, these are to the left of $s = 2$.

First we will prove that

$$(4m)^{-s+u+5/2} N^{-s+4+k/2} Z^{\pm}(u, s)\, T\mathscr{F}^{\pm}(u, s, y_2) =$$

$$\sum_{\substack{D\in Z\\ \pm D>0}} \frac{1}{N} \int_0^\infty \int_{-\infty}^\infty \int_0^N TC_1\left(\begin{pmatrix} 1 & x^2 & & \\ & 1 & & \\ & & 1 & \\ & & -x_2 & 1 \end{pmatrix}\begin{pmatrix} y_1y_2 & & & \\ & y_1 & & \\ & & y_2^{-1} & \\ & z & & 1 \end{pmatrix}; U_1(-ND), v\right)$$

$$e\left(\frac{-iDy_1}{4m(1+iz)}\right) y_1^{u-3/2} \sqrt{1+iz}\; e(-N^{-1}x_2)\, dx_2\, dz \frac{dy_1}{y_1}\,, \tag{8.3}$$

where the coefficients C_1 are as in Proposition 2.2, and $\mathscr{F}^{\pm}(u, s, y_2)$ is defined in Section 3. Let Δ_z denote $\sqrt{z^2+1}$. Since

$$\begin{pmatrix} y_1y_2 & & & \\ & y_1 & & \\ & & y_2^{-1} & \\ & z & & 1 \end{pmatrix} = \begin{pmatrix} y_1y_2 & & & \\ & \Delta_z^{-1}y_1 & & \Delta_z^{-1}y_1z \\ & & y_2^{-1} & \\ & & & \Delta_z \end{pmatrix}\kappa_z,$$

where κ_z is defined by (3.30), and since $C_1\left(\begin{pmatrix} E & X \\ & E \end{pmatrix} g;\, U,\, v\right) = C_1(g;\, U,\, v)$ for real symmetric matrices X, the right hand side of (8.3) equals

$$\sum_{\substack{D\in\mathbf{Z}\\ \pm D>0}} \int_0^\infty \int_{-\infty}^\infty T\left\{ \frac{1}{N}\int_0^N C_1\left(\begin{pmatrix} 1 & x_2 & & \\ & 1 & & \\ & & 1 & \\ & & -x_2 & 1 \end{pmatrix}\begin{pmatrix} y_1y_2 & & & \\ & \Delta_z^{-1}y_1 & & \\ & & y_2^{-1} & \\ & & & \Delta_z \end{pmatrix};\right.\right.$$

$$\left.\left. U_1(-ND),\, v\right) e(-N^{-1}x_2)\, dx_2 \sigma(\kappa_z)\right\} e\left(\frac{-iDy_1}{4m(1+iz)}\right) y_1^{u-3/2}\sqrt{1+iz}\; dz \frac{dy_1}{y_1}. \tag{8.4}$$

Observe that by (2.6), this is equal to zero unless $D \equiv r^2 \bmod 4m/N$. Now by Proposition 6.1, and the invariance of C_1 under scalar matrices, we have

$$\frac{1}{N}\int_0^N C_1\left(\begin{pmatrix} 1 & x_2 & & \\ & 1 & & \\ & & 1 & \\ & & -x_2 & 1 \end{pmatrix}\begin{pmatrix} y_1y_2 & & & \\ & \Delta_z^{-1}y_1 & & \\ & & y_2^{-1} & \\ & & & \Delta_z \end{pmatrix};\, U_1(-ND),\, v\right)$$

$$e(-N^{-1}x_2)\, dx_2 = \left(\frac{|D|}{4mN}\right)^{s-4} N^{k/2} e\left(\frac{iy_1D}{4m\Delta_z^2}\right) \mathscr{L}(s, D)\, W^{\pm}\left(\frac{|D|y_1}{4m\Delta_z^2}, N^{-1}\Delta_z y_2;\, s\right),$$

where the sign $\pm$ is $+$ if $D>0$, $-$ if $D<0$. Substituting this expression into the right hand side of (8.4), replacing y_1 by $4my_1/|D|$ and applying the definition of $\mathscr{F}^{\pm}(u, s, y_2)$, we obtain the left hand side of (8.3). Thus we have proved formula (8.3).

Next we prove that

$$\sum_{\substack{D\in\mathbf{Z}\\ D\neq 0}} \frac{1}{N}\int_{-\infty}^\infty \int_0^N C_1\left(\begin{pmatrix} 1 & x_2 & & \\ & 1 & & \\ & & 1 & \\ & & -x_2 & 1 \end{pmatrix}\begin{pmatrix} y_1y_2 & & & \\ & y_1 & & \\ & & y_2^{-1} & \\ & z & & 1 \end{pmatrix};\, U_1(-ND),\, v\right)$$

$$e\left(\frac{-iDy_1}{4m(1+iz)}\right) y_1^{-1}\sqrt{1+iz}\, e(-N^{-1}x_2)dx_2\, dz =$$

$$\frac{1}{N^2}\int_0^1\int_0^N\int_0^N \mathscr{E}_1\left(\begin{pmatrix} 1 & x_2 & & x_4 \\ & 1 & & \\ & & 1 & \\ & z & -x_2 & 1 \end{pmatrix}\begin{pmatrix} y_1y_2 & & & \\ & y_1 & & \\ & & y_2^{-1} & \\ & & & 1 \end{pmatrix};\, v\right)$$

$$\times \sqrt{1+iy_1z}\, e(-N^{-1}x_2)\, dx_2\, dx_4\, dz$$

$$-\frac{1}{N}\int_{-\infty}^{\infty}\int_0^N C_1\left(\begin{pmatrix}1 & x_2 & & \\ & 1 & & \\ & & 1 & \\ & & -x_2 & 1\end{pmatrix}\begin{pmatrix}y_1y_2 & & & \\ & y_1 & & \\ & & y_2^{-1} & \\ & z & & 1\end{pmatrix};v\right)$$

$$\times y_1^{-1}\sqrt{1+iz}\,e(-N^{-1}x_2)\,dx_2\,dz\,. \tag{8.5}$$

Let

$$H_1(x_3)=\frac{1}{N^2}\int_0^N\int_0^N \mathscr{E}_1\left(\begin{pmatrix}1 & & x_4 & x_3\\ & 1 & x_3 & \\ & & 1 & \\ & & & 1\end{pmatrix}\begin{pmatrix}1 & x_2 & & \\ & 1 & & \\ & & 1 & \\ & & -x_2 & 1\end{pmatrix}g;v\right)$$

$$\times e(-N^{-1}x_2)\,dx_2\,dx_4\,.$$

By (2.19), we have $H_1(x_3-n)=H_1(x_3)$ if $n\in N\mathbf{Z}$, so by Fourier inversion,

$$H_1(0)=\sum_{n\in\mathrm{N}}\frac{1}{N}\int_0^N H_1(x_3)\,e(N^{-1}nx_3)\,dx_3\,.$$

Substituting the definition of H_1, replacing x_2 by x_2+nx_3 and applying a matrix identity, this becomes

$$\frac{1}{N^3}\sum_{n\in Z}\int_0^N\int_0^N\int_0^N \mathscr{E}_1\left(\begin{pmatrix}1 & & & \\ & 1 & & \\ & & 1 & \\ & -n & & 1\end{pmatrix}\begin{pmatrix}1 & & x_4 & x_3\\ & 1 & x_3 & \\ & & 1 & \\ & & & 1\end{pmatrix}\right.$$

$$\left.\times\begin{pmatrix}1 & x_2 & & \\ & 1 & & \\ & & 1 & \\ & & -x_2 & 1\end{pmatrix}\begin{pmatrix}1 & & & \\ & 1 & & \\ & & 1 & \\ & n & & 1\end{pmatrix}g;v\right)e(-N^{-1}x_2)\,dx_2\,dx_3\,dx_4\,.$$

Now if

$$g'=\begin{pmatrix}1 & & x_4 & x_3\\ & 1 & x_3 & \\ & & 1 & \\ & & & 1\end{pmatrix}\begin{pmatrix}1 & x_2 & & \\ & 1 & & \\ & & 1 & \\ & & -x_2 & 1\end{pmatrix}\begin{pmatrix}1 & & & \\ & 1 & & \\ & & 1 & \\ & n & & 1\end{pmatrix}g\,,$$

and if

$$Z_g=\begin{pmatrix}Z_{11}(g) & Z_{12}(g)\\ Z_{12}(g) & Z_{22}(g)\end{pmatrix},$$

then it may be checked that

$$\det(E+U_1(-n)Z_{g'})=(1+nZ_{22}(g))^{-1}\,,$$

independent of the values of x_2, x_3 and x_4. Thus by (2.21), we have

$$\frac{1}{N^2}\int_0^N\int_0^N \mathscr{E}_1\left(\begin{pmatrix}1 & x_2 & x_4 & \\ & 1 & & \\ & & 1 & \\ & & -x_2 & 1\end{pmatrix} g;\, v\right) e(-N^{-1}x_2)\,dx_2\,dx_4 = H_1(0)$$

$$= \sum_{n\in\mathbf{Z}} \sqrt{1 + nZ_{22}(g)}\, \times$$

$$\frac{1}{N^3}\int_0^N\int_0^N\int_0^N \mathscr{E}_1\left(\begin{pmatrix}1 & & x_4 & x_3\\ & 1 & x_3 & \\ & & 1 & \\ & & & 1\end{pmatrix}\begin{pmatrix}1 & x_2 & & \\ & 1 & & \\ & & 1 & \\ & & -x_2 & 1\end{pmatrix}\begin{pmatrix}1 & & & \\ & 1 & & \\ & & 1 & \\ & n & & 1\end{pmatrix} g;\, v\right)$$

$$\times\, e(-N^{-1}x_2)\,dx_2\,dx_3\,dx_4\,.$$

We will now apply this with

$$g = \begin{pmatrix}1 & & & \\ & 1 & & \\ & & 1 & \\ & z & & 1\end{pmatrix}\begin{pmatrix}y_1y_2 & & & \\ & y_1 & & \\ & & y_2^{-1} & \\ & & & 1\end{pmatrix}, \text{ so that } Z_g = \begin{pmatrix} iy_1y_2^2 & \\ & \dfrac{iy_1}{1+iy_1z}\end{pmatrix}.$$

Thus

$$1 + nZ_{22}(g) = \frac{1 + iy_1(n+z)}{1 + iy_1z},$$

and

$$\sqrt{1 + iy_1z}\,\frac{1}{N^2}\int_0^N\int_0^N \mathscr{E}_1\left(\begin{pmatrix}1 & x_2 & x_4 & \\ & 1 & & \\ & & 1 & \\ & z & -x_2 & 1\end{pmatrix}\begin{pmatrix}y_1y_2 & & & \\ & y_1 & & \\ & & y_2^{-1} & \\ & & & 1\end{pmatrix};v\right)$$

$$\times\, e(-N^{-1}x_2)\,dx_2\,dx_4 = \sum_{n\in\mathbf{Z}} \sqrt{1 + iy_1(n+z)}\, \times$$

$$\frac{1}{N^3}\int_0^N\int_0^N\int_0^N \mathscr{E}_1\left(\begin{pmatrix}1 & & x_4 & x_3\\ & 1 & x_3 & \\ & & 1 & \\ & & & 1\end{pmatrix}\begin{pmatrix}1 & x_2 & & \\ & 1 & & \\ & & 1 & \\ & & -x_2 & 1\end{pmatrix}\right.$$

$$\left.\times \begin{pmatrix}1 & & & \\ & 1 & & \\ & & 1 & \\ & n+z & & 1\end{pmatrix}\begin{pmatrix}y_1y_2 & & & \\ & y_1 & & \\ & & y_2^{-1} & \\ & & & 1\end{pmatrix};\, v\right) e(-N^{-1}x_2)\,dx_2\,dx_3\,dx_4\,.$$

Now integrating z from 0 to 1, and collapsing the sum over n, we obtain

$$\frac{1}{N^2}\int_0^1\int_0^N\int_0^N \mathscr{E}_1\left(\begin{pmatrix}1 & x_2 & & x_4\\ & 1 & & \\ & & 1 & \\ & z & -x_2 & 1\end{pmatrix}\begin{pmatrix}y_1y_2 & & & \\ & y_1 & & \\ & & y_2^{-1} & \\ & & & 1\end{pmatrix};v\right)$$
$$\times\sqrt{1+iy_1z}\,e(-N^{-1}x_2)\,dx_2\,dx_4\,dz =$$
$$\frac{1}{N^3}\int_{-\infty}^{\infty}\int_0^N\int_0^N\int_0^N \mathscr{E}_1\left(\begin{pmatrix}1 & & x_4 & x_3\\ & 1 & x_3 & \\ & & 1 & \\ & & & 1\end{pmatrix}\begin{pmatrix}1 & x_2 & & \\ & 1 & & \\ & & 1 & \\ & & -x_2 & 1\end{pmatrix}\right.$$
$$\left.\begin{pmatrix}1 & & & \\ & 1 & & \\ & & 1 & \\ & z & & 1\end{pmatrix}\begin{pmatrix}y_1y_2 & & & \\ & y_1 & & \\ & & y_2^{-1} & \\ & & & 1\end{pmatrix};v\right)$$
$$\times\sqrt{1+iy_1z}\,e(-N^{-1}x_2)\,dx_2\,dx_3\,dx_4\,dz\,. \tag{8.6}$$

Replacing z by $y_1^{-1}z$ shows that (8.6) equals $H_2(0)$, where for fixed y_1, y_2 we have

$$H_2(x_1)=\frac{1}{N^3}\int_{-\infty}^{\infty}\int_0^N\int_0^N\int_0^N \mathscr{E}_1\left(\begin{pmatrix}1 & & x_4 & x_3\\ & 1 & x_3 & x_1\\ & & 1 & \\ & & & 1\end{pmatrix}\begin{pmatrix}1 & x_2 & & \\ & 1 & & \\ & & 1 & \\ & & -x_2 & 1\end{pmatrix}\right.$$
$$\left.\begin{pmatrix}y_1y_2 & & & \\ & y_1 & & \\ & & y_2^{-1} & \\ & z & & 1\end{pmatrix};v\right)y_1^{-1}\sqrt{1+iz}\,e(-N^{-1}x_2)\,dx_2\,dx_3\,dx_4\,dz\,.$$

By (2.19), we have $H_2(x_1+n)=H_2(x_1)$ if $n\in 4m\mathbf{Z}$, and so by Fourier inversion, we have

$$H_2(0)=\sum_{D\in\mathbf{Z}}\frac{1}{4m}\int_0^{4m}H_2(x_1)\,e((4m)^{-1}Dx_1)\,dx_1\,.$$

By (2.5), this equals

$$\sum_{D\in\mathbf{Z}}\frac{1}{4mN^3}\int_{-\infty}^{\infty}\int_0^N\int_0^N\int_0^N\int_0^{4m} \mathscr{E}_1\left(\begin{pmatrix}1 & & x_4 & x_3\\ & 1 & x_3 & x_1\\ & & 1 & \\ & & & 1\end{pmatrix}\begin{pmatrix}1 & x_2 & & \\ & 1 & & \\ & & 1 & \\ & & -x_2 & 1\end{pmatrix}\right.$$
$$\left.\times\begin{pmatrix}y_1y_2^{-1} & & & \\ & y_1 & & \\ & & y_2^{-1} & \\ & z & & 1\end{pmatrix};v\right)$$
$$y_1^{-1}\sqrt{1+iz}\,e((4m)^{-1}Dx_1-N^{-1}x_2)\,dx_1\,dx_2\,dx_3\,dx_4\,dz$$

$$= \sum_{D \in \mathbf{Z}} \frac{1}{N} \int_{-\infty}^{\infty} \int_{0}^{N} C_1 \left(\begin{pmatrix} 1 & x_2 & & \\ & 1 & & \\ & & 1 & \\ & & -x_2 & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2 & & & \\ & y_1 & & \\ & & y_2^{-1} & \\ & z & & 1 \end{pmatrix} ; U_1(-ND), v \right)$$

$$\times\, e\left(\frac{-iDy_1}{4m(1+iz)} \right) y_1^{-1} \sqrt{1+iz}\, e(-N^{-1} x_2)\, dx_2\, dz\,.$$

This completes the proof of (8.5).

Similarly, we will prove

$$\sum_{\substack{D \in \mathbf{Z} \\ D \neq 0}} \frac{1}{N} \int_{-\infty}^{\infty} \int_{0}^{N} \sum_{\mu \bmod 2m/N} e\left(\frac{N}{2m} {}^{\mathsf{T}} v \mu \right) C_0 \left(\begin{pmatrix} 1 & & & \\ -x_2 & 1 & & \\ & & 1 & x_2 \\ & & & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2^{-1} & & & \\ & & y_1 & \\ z y_2 & & & y_2 \\ & & & & 1 \end{pmatrix} \right.$$

$$\left. \times\, J; U_0(-4mD), \mu \right) y_1^{-1} y_2^3 \sqrt{1+iz}\, e\left(\frac{-iDy_1}{y_2^2(1+iz)} \right) e(-N^{-1} x_2)\, dx_2\, dz =$$

$$\frac{1}{N} \int_0^N \int_0^1 \int_0^N \sum_{\mu \bmod 2m/N} e\left(\frac{N}{2m} {}^{\mathsf{T}} v \mu \right) \mathscr{E}_0 \left(\begin{pmatrix} 1 & & & \\ -x_2 & 1 & & x_4 \\ z & & 1 & x_2 \\ & & & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2^{-1} & & & \\ & y_1 & & \\ & & y_2 & \\ & & & 1 \end{pmatrix} J; \mu \right)$$

$$y_2 \sqrt{1 + i y_1 y_2^{-1} z}\, e(-N^{-1} x_2)\, dx_2\, dx_4\, dz$$

$$-\frac{1}{N} \int_{-\infty}^{\infty} \int_{0}^{N} \sum_{\mu \bmod 2m/N} e\left(\frac{N}{2m} {}^{\mathsf{T}} v \mu \right) C_0 \left(\begin{pmatrix} 1 & & & \\ -x_2 & 1 & & \\ & & 1 & x_2 \\ & & & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2^{-1} & & & \\ & & y_1 & \\ z y_2 & & & y_2 \\ & & & & 1 \end{pmatrix} \right.$$

$$\left. \times\, J; 0, \mu \right) y_1^{-1} y_2^3 \sqrt{1+iz}\, e(-N^{-1} x_2)\, dx_2\, dz\,. \tag{8.7}$$

Let

$H_3(x_3) =$

$$\frac{1}{N} \int_0^1 \int_0^N {\sum_{\mu \bmod 2m/N}}' e\left(\frac{N}{2m} {}^{\mathsf{T}} v \mu \right) \mathscr{E}_0 \left(\begin{pmatrix} 1 & & & x_3 \\ & 1 & x_3 & x_4 \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ -x_2 & 1 & & \\ & & 1 & x_2 \\ & & & 1 \end{pmatrix} g; \mu \right)$$

$$e(-N^{-1} x_2)\, dx_2\, dx_4\,.$$

Note that in this definition, the integrand is periodic as a function of x_2, with period N, by (2.18). By (2.20), we have $H_3(x_3 + n) = H_3(x_3)$ if $n \in \mathbf{Z}$. Consequently,

$$H_3(0) = \sum_{n \in \mathbf{Z}} \int_0^1 H_3(x_3)\, e(-n x_3)\, dx_3\,.$$

Substituting the definition of H_3, replacing x_2 by $x_2 - Nnx_3$, and applying a matrix identity, this becomes

$$\sum_{n\in\mathbf{Z}} \frac{1}{N}\int_0^1\int_0^1\int_0^N \sum_{\mu \bmod 2m/N} e\left(\frac{N}{2m}{}^{\mathsf{T}}\nu\mu\right) \times$$

$$\mathscr{E}_0\left(\begin{pmatrix} 1 & & & \\ & 1 & & \\ -Nn & & 1 & \\ & & & 1\end{pmatrix}\begin{pmatrix} 1 & & & x_3 \\ & 1 & x_3 & x_4 \\ & & 1 & \\ & & & 1\end{pmatrix}\begin{pmatrix} 1 & & & \\ -x_2 & 1 & & \\ & & 1 & x_2 \\ & & & 1\end{pmatrix}\right.$$

$$\left.\begin{pmatrix} 1 & & & \\ & 1 & & \\ Nn & & 1 & \\ & & & 1\end{pmatrix} g;\mu\right) e(-N^{-1}x_2)\,dx_2\,dx_3\,dx_4\,.$$

Now by (2.22), this equals

$$\sum_{n\in\mathbf{Z}} \sqrt{1 + NnZ_{11}(g)}\,\frac{1}{N}\int_0^1\int_0^1\int_0^N \sum_{\mu \bmod 2m/N} e\left(\frac{N}{2m}{}^{\mathsf{T}}\nu\mu\right) \times$$

$$\mathscr{E}_0\left(\begin{pmatrix} 1 & & & x_3 \\ & 1 & x_3 & x_4 \\ & & 1 & \\ & & & 1\end{pmatrix}\begin{pmatrix} 1 & & & \\ -x_2 & 1 & & \\ & & 1 & x_2 \\ & & & 1\end{pmatrix}\begin{pmatrix} 1 & & & \\ & 1 & & \\ Nn & & 1 & \\ & & & 1\end{pmatrix} g;\mu\right)$$

$$e(-N^{-1}x_2)\,dx_2\,dx_3\,dx_4\,.$$

We now apply this with

$$g = \begin{pmatrix} 1 & & & \\ & 1 & & \\ z & & 1 & \\ & & & 1\end{pmatrix}\begin{pmatrix} y_1y_2^{-1} & & & \\ & y_1 & & \\ & & y_2 & \\ & & & 1\end{pmatrix} J \quad \text{so that } Z_g = \begin{pmatrix} \dfrac{iy_1y_2^{-2}}{1 + iy_1y_2^{-2}z} & \\ & iy_1\end{pmatrix}.$$

Thus

$$1 + NnZ_{11}(g) = \frac{1 + iy_1y_2^{-2}(z + Nn)}{1 + iy_1y_2^{-2}z}\,.$$

Therefore, we have proved that

$$\sqrt{1 + iy_1y_2^{-2}z}\,\frac{1}{N}\int_0^1\int_0^N \sum_{\mu \bmod 2m/N} e\left(\frac{N}{2m}{}^{\mathsf{T}}\nu\mu\right)$$

$$\mathscr{E}_0\left(\begin{pmatrix} 1 & & & \\ -x_2 & 1 & & x_4 \\ z & & 1 & x_2 \\ & & & 1\end{pmatrix}\begin{pmatrix} y_1y_2^{-1} & & & \\ & y_1 & & \\ & & y_2 & \\ & & & 1\end{pmatrix} J;\mu\right) e(-N^{-1}x_2)\,dx_2\,dx_4 =$$

$$\sum_{n\in\mathbf{Z}} \sqrt{1+iy_1y_2^{-2}(z+Nn)}\,\frac{1}{N}\int_0^1\int_0^1\int_0^N \sum_{\mu \bmod 2m/N} e\left(\frac{N}{2m}{}^{\mathsf{T}}v\mu\right)\times$$

$$\mathscr{E}_0\left(\left(\begin{pmatrix}1&&&x_3\\&1&x_3&x_4\\&&1&\\&&&1\end{pmatrix}\begin{pmatrix}1&&&\\-x_2&1&&\\&&1&x_2\\&&&1\end{pmatrix}\right)\left(\begin{pmatrix}1&&&\\&1&&\\z+Nn&&1&\\&&&1\end{pmatrix}\right.\right.$$

$$\left.\left.\begin{pmatrix}y_1y_2^{-1}&&&\\&y_1&&\\&&y_2&\\&&&1\end{pmatrix}J;\,\mu\right)e(-N^{-1}x_2)\,dx_2\,dx_3\,dx_4\,.$$

Integrating z from 0 to N, and collapsing the sum over n, we thus obtain

$$\frac{1}{N}\int_0^N\int_0^1\int_0^N \sum_{\mu \bmod 2m/N} e\left(\frac{N}{2m}{}^{\mathsf{T}}v\mu\right)\times$$

$$\mathscr{E}_0\left(\begin{pmatrix}1&&&\\-x_2&1&&x_4\\z&&1&x_2\\&&&1\end{pmatrix}\begin{pmatrix}y_1y_2^{-1}&&&\\&y_1&&\\&&y_2&\\&&&1\end{pmatrix}J;\,\mu\right)$$

$$\times\, y_2\sqrt{1+iy_1y_2^{-2}z}\,e(-N^{-1}x_2)\,dx_2\,dx_4\,dz =$$

$$\frac{1}{N}\int_{-\infty}^{\infty}\int_0^1\int_0^1\int_0^N \sum_{\mu \bmod 2m/N} e\left(\frac{N}{2m}{}^{\mathsf{T}}v\mu\right)\times$$

$$\mathscr{E}_0\left(\begin{pmatrix}1&&&x_3\\&1&x_3&x_4\\&&1&\\&&&1\end{pmatrix}\begin{pmatrix}1&&&\\-x_2&1&&\\&&1&x_2\\&&&1\end{pmatrix}\begin{pmatrix}1&&&\\&1&&\\z&&1&\\&&&1\end{pmatrix}\right.$$

$$\left.\begin{pmatrix}y_1y_2^{-1}&&&\\&y_1&&\\&&y_2&\\&&&1\end{pmatrix}J;\,\mu\right)\,y_2\sqrt{1+iy_1y_2^{-2}z}\,e(-N^{-1}x_2)\,dx_2\,dx_3\,dx_4\,dz\,. \quad (8.8)$$

Replacing z by $y_1^{-1}y_2^2z$ shows that (8.8) equals $H_4(0)$, where

$$H_4(x_1) = \frac{1}{N}\int_{-\infty}^{\infty}\int_0^1\int_0^1\int_0^N \sum_{\mu \bmod 2m/N} e\left(\frac{N}{2m}{}^{\mathsf{T}}v\mu\right)\times$$

$$\mathscr{E}_0\left(\begin{pmatrix}1&&x_1&x_3\\&1&x_3&x_4\\&&1&\\&&&1\end{pmatrix}\begin{pmatrix}1&&&\\-x_2&1&&\\&&1&x_2\\&&&1\end{pmatrix}\begin{pmatrix}y_1y_2^{-1}&&&\\&y_1&&\\zy_2&&y_2&\\&&&1\end{pmatrix}J;\,\mu\right)$$

$$\times\, y_1^{-1}y_2^3\sqrt{1+iz}\,e(-N^{-1}\,x_2)\,dx_2\,dx_3\,dx_4\,dz\,.$$

It follows from (2.20) that $H_4(x_1+n)=H_4(x_1)$ for $n\in\mathbf{Z}$. Therefore

$$H_4(0)=\sum_{D\in\mathbf{Z}}\int_0^1 H_4(x_1)\,e(Dx_1)\,dx_1\,.$$

By (2.5), this equals

$$\sum_{D\in\mathbf{Z}}\frac{1}{N}\int_{-\infty}^{\infty}\int_0^1\int_0^1\int_0^N\int_0^1\sum_{\mu \bmod 2m/N} e\left(\frac{N}{2m}{}^{\mathrm{T}}\nu\mu\right)\times$$

$$\mathscr{E}_0\left(\left(\begin{pmatrix}1 & & x_1 & x_3\\ & 1 & x_3 & x_4\\ & & 1 & \\ & & & 1\end{pmatrix}\begin{pmatrix}1 & & & \\ -x_2 & 1 & & \\ & & 1 & x_2\\ & & & 1\end{pmatrix}\begin{pmatrix}y_1y_2^{-1} & & & \\ & & y_1 & \\ zy_2 & & & y_2\\ & & & & 1\end{pmatrix}\right)J;\mu\right)$$

$$y_1^{-1}y_2^3\sqrt{1+iz}\,e(Dx_1-N^{-1}x_2)\,dx_1\,dx_2\,dx_3\,dx_4\,dz=$$

$$\sum_{D\in\mathbf{Z}}\frac{1}{N}\int_{-\infty}^{\infty}\int_0^N\sum_{\mu \bmod 2m/N} e\left(\frac{N}{2m}{}^{\mathrm{T}}\nu\mu\right)C_0\left(\left(\begin{pmatrix}1 & & & \\ -x_2 & 1 & & \\ & & 1 & x_2\\ & & & 1\end{pmatrix}\begin{pmatrix}y_1y_2^{-1} & & & \\ & & y_1 & \\ zy_2 & & & y_2\\ & & & & 1\end{pmatrix}\right.\right.$$

$$\left.\times J;\,U_0(-4mD),\,u\right)\,y_1^{-1}y_2^3\sqrt{1+iz}\,e\left(\frac{-iDy_1}{y_2^2(1+iz)}\right)e(-N^{-1}x_2)\,dx_2dz\,.$$

Thus by (8.8), we have proved (8.7).

Next we have

$$\frac{1}{N^2}\int_0^1\int_0^N\int_0^N\mathscr{E}_1\left(\left(\begin{pmatrix}1 & x_2 & & x_4\\ & 1 & & \\ & & 1 & \\ & z & -x_2 & 1\end{pmatrix}\begin{pmatrix}y_1y_2 & & & \\ & y_1 & & \\ & & y_2^{-1} & \\ & & & 1\end{pmatrix}\right);\nu\right)$$

$$\sqrt{1+iy_1z}\,e(-N^{-1}x_2)\,dx_2\,dx_4\,dz=$$

$$\frac{1}{2mN^2}\int_0^N\int_0^1\int_0^N\sum_{\mu \bmod 2m/N} e\left(-\frac{N}{2m}{}^{\mathrm{T}}\nu\mu\right)\mathscr{E}_0\left(\left(\begin{pmatrix}1 & & & \\ -x_2 & 1 & & x_4\\ z & & 1 & x_2\\ & & & 1\end{pmatrix}\right.\right.\times$$

$$\left.\left.\begin{pmatrix}y_1^{-1}y_2^{-1} & & & \\ & y_1^{-1} & & \\ & & y_2 & \\ & & & 1\end{pmatrix}\right)J;\,\mu\right)\,y_1y_2\sqrt{1+iy_1^{-1}y_2^{-2}z}\,e(-N^{-1}x_2)\,dx_2\,dx_4\,dz \qquad (8.9)$$

Indeed, to prove this one applies to the left side of (8.9) the identity (2.7), the invariance of $\mathscr{E}_0$ by scalar matrices, and interchanges the roles of x_4 and $-z$.

Now (8.1) equals $I_1 + I_2$, where I_1 *and* I_2 are the respective contributions to (8.3) from $1 < y_1 < \infty$ and from $0 < y_1 < 1$, respectively. Thus

$$I_1 = \sum_{\substack{D \in \mathbf{Z} \\ D \neq 0}} \frac{1}{N} \int_1^\infty \int_{-\infty}^\infty \int_0^N TC_1 \left(\begin{pmatrix} 1 & x_2 & & \\ & 1 & & \\ & & 1 & \\ & & -x_2 & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2 & & & \\ & y_1 & & \\ & & y_2^{-1} & \\ & z & & 1 \end{pmatrix}; \right.$$

$$\left. \times\, U_1(-ND), v \right) e\left(\frac{-iDy_1}{4m(1+iz)} \right) y_1^{u-3/2} \sqrt{1+iz}\, e(-N^{-1}x_2)\, dx_2 dz \frac{dy_1}{y_1}, \quad (8.10)$$

while

$$I_2 = \sum_{\substack{D \in \mathbf{Z} \\ D \neq 0}} \frac{1}{N} \int_0^1 \int_{-\infty}^\infty \int_0^N TC_1 \left(\begin{pmatrix} 1 & x_2 & & \\ & 1 & & \\ & & 1 & \\ & & -x_2 & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2 & & & \\ & y_1 & & \\ & & y_2^{-1} & \\ & z & & 1 \end{pmatrix}; U_1(-ND), v \right)$$

$$\times\, e\left(\frac{-iDy_1}{4m(1+iz)} \right) y_1^{u-3/2} \sqrt{1+iz}\, e(-N^{-1}x_2)\, dx_2 dz \frac{dy_1}{y_1}. \quad (8.11)$$

Now I_1 is analytic for all u and s under consideration. To see this, use Proposition 6.1 to express the integral (8.10) in terms of the Whittaker functions and the Dirichlet series $\mathscr{L}(s, D)$. Then because of the at most polynomial growth of the coefficients (cf. the Remark following the statement of Proposition 7.1) and the rapid decay of the Whittaker functions (Proposition 3.6), the sum (8.10) is very rapidly convergent.

The meromorphic continuation of I_2 requires some work. By (8.5), $I_2 = I_3 - I_4$, where

$$I_3 = \frac{1}{N^2} \int_0^1 \int_0^1 \int_0^N \int_0^N \mathscr{E}_1 \left(\begin{pmatrix} 1 & x_2 & x_4 & \\ & 1 & & \\ & & 1 & \\ z & & -x_2 & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2 & & & \\ & y_1 & & \\ & & y_2^{-1} & \\ & & & 1 \end{pmatrix}; v \right)$$

$$\times\, y_1^{u-1/2} \sqrt{1+iy_1 z}\, e(-N^{-1}x_2)\, dx_2\, dx_4\, dz \frac{dy_1}{y_1}$$

while

$$I_4 = \frac{1}{N} \int_0^1 \int_{-\infty}^\infty \int_0^N TC_1 \left(\begin{pmatrix} 1 & x_2 & & \\ & 1 & & \\ & & 1 & \\ & & -x_2 & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2 & & & \\ & y_1 & & \\ & & y_2^{-1} & \\ & z & & 1 \end{pmatrix}; 0, v \right)$$

$$\times\, y_1^{u-3/2} \sqrt{1+iz}\, e(-N^{-1}x_2)\, dx_2 dz \frac{dy_1}{y_1}.$$

Proceeding as in the proof of (8.3), I_4 equals

$$\int_0^1 \int_{-\infty}^{\infty} T\left\{ \frac{1}{N}\int_0^N C_1\left(\begin{pmatrix} 1 & x_2 & & \\ & 1 & & \\ & & 1 & \\ & & -x_2 & 1 \end{pmatrix}\right.\right.$$

$$\left.\left.\begin{pmatrix} y_1 y_2 & & & \\ & \Delta_z^{-1} y_1 & & \\ & & y_2^{-1} & \\ & & & \Delta_z \end{pmatrix}; 0, v\right) e(-N^{-1}x_2)\,dx_2\,\sigma(\kappa_z)\right\} y_1^{u-3/2}\sqrt{1+iz}\,dz\,\frac{dy_1}{y_1}.$$

Thus by Proposition 6.1, this equals

$$N^{-s+4+k/2}\mathscr{L}(s,0)\int_0^1\int_{-\infty}^{\infty} T(W^0(\Delta_z^{-2}y_1, N^{-1}\Delta_z y_2; s)\,\sigma(\kappa_z))\,y_1^{u-3/2}\sqrt{1+iz}\,dz\,\frac{dy_1}{y_1}.$$

Thus by (3.54), we have

$$I_4 = (u-s+5/2)^{-1} N^{-s+4+k/2}\mathscr{L}(s,0)$$

$$\int_{-\infty}^{\infty} \Delta_z^{2s-8}\, T(W^0(1, N^{-1}\Delta_z y_2; s)\,\sigma(\kappa_z))\sqrt{1+iz}\,dz\,. \tag{8.12}$$

On the other hand, consider I_3. By (8.9), this equals

$$\frac{1}{2mN^2}\int_0^1\int_0^N\int_0^1\int_0^N \sum_{\mu \bmod 2m/N} e\left(-\frac{N}{2m}{}^{\mathsf{T}}v\mu\right)$$

$$\times\, T\mathscr{E}_0\left(\begin{pmatrix} 1 & & & \\ -x_2 & 1 & & x_4 \\ z & & 1 & x_2 \\ & & & 1 \end{pmatrix}\begin{pmatrix} y_1^{-1}y_2^{-1} & & & \\ & y_1^{-1} & & \\ & & y_2 & \\ & & & 1 \end{pmatrix} J; \mu\right)$$

$$\times\, y_1^{u+1/2} y_2\sqrt{1+iy_1^{-1}y_2^{-2}z}\; e(-N^{-1}x_2)\,dx_2\,dx_4\,dz\,\frac{dy_1}{y_1},$$

and substituting y_1^{-1} for y_1, this equals

$$\frac{1}{2mN^2}\int_1^\infty\int_0^N\int_0^1\int_0^N \sum_{\mu \bmod 2m/N} e\left(-\frac{N}{2m}{}^{\mathsf{T}}v\mu\right)$$

$$\times\, T\mathscr{E}_0\left(\begin{pmatrix} 1 & & & \\ -x_2 & 1 & & x_4 \\ z & & 1 & x_2 \\ & & & 1 \end{pmatrix}\begin{pmatrix} y_1 y_2^{-1} & & & \\ & y_1 & & \\ & & y_2 & \\ & & & 1 \end{pmatrix} J; \mu\right)$$

$$\times\, y_1^{-u-1/2} y_2\sqrt{1+iy_1y_2^{-2}z}\; e(-N^{-1}x_2)\,dx_2\,dx_4\,dz\,\frac{dy_1}{y_1}.$$

By (8.7), this equals $I_5 + I_6$, where I_5 equals

$$\sum_{\substack{D \in \mathbf{Z} \\ D \neq 0}} \frac{1}{N} \int_1^{\infty} \int_{-\infty}^{\infty} \int_0^N \sum_{\mu \bmod 2m/N} e\left(\frac{N}{2m} {}^{\mathrm{T}}\nu\mu\right)$$

$$\times\, TC_0\left(\left(\begin{pmatrix} 1 & & & \\ -x_2 & 1 & & \\ & & 1 & x_2 \\ & & & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2^{-1} & & & \\ & y_1 & & \\ z & & y_2 & \\ & & & 1 \end{pmatrix} J;\ U_0(-4mD),\ \mu\right)\right)$$

$$\times\, y_1^{-u-3/2}\, y_2^2 \sqrt{1 + iy_2^{-1} z}\; e\left(\frac{-iDy_1 y_2^{-1}}{y_2 + iz}\right) e(-N^{-1} x_2)\, dx_2\, dz \frac{dy_1}{y_1},$$

while I_6 equals

$$\frac{1}{N} \int_1^{\infty} \int_{-\infty}^{\infty} \int_0^N \sum_{\mu \bmod 2m/N} e\left(\frac{N}{2m} {}^{\mathrm{T}}\nu\mu\right)$$

$$\times\, TC_0\left(\left(\begin{pmatrix} 1 & & & \\ -x_2 & 1 & & \\ & & 1 & x_2 \\ & & & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2^{-1} & & & \\ & y_1 & & \\ zy_2 & & y_2 & \\ & & & 1 \end{pmatrix} J;\ 0,\ \mu\right)\right)$$

$$\times\, y_1^{-u-3/2}\, y_2^3 \sqrt{1 + iz}\; e(-N^{-1} x_2)\, dx_2\, dz \frac{dy_1}{y_1}.$$

Now I_5 is convergent for all values of s and u under consideration. (The reason is the same as that for I_1, except that one uses Proposition 6.2.) However, I_6 has poles, which we must consider. This equals

$$\int_1^{\infty} \int_{-\infty}^{\infty} T\left\{ \sum_{\mu \bmod 2m/N} e\left(\frac{N}{2m} {}^{\mathrm{T}}\nu\mu\right) \frac{1}{N} \int_0^N C_0\left(\left(\begin{pmatrix} 1 & & & \\ -x_2 & 1 & & \\ & & 1 & x_2 \\ & & & 1 \end{pmatrix}\right.\right.\right.$$

$$\left.\left.\left.\begin{pmatrix} y_1 y_2^{-1} \Delta_z^{-1} & & & \\ & y_1 & & \\ & & \Delta_z y_2 & \\ & & & 1 \end{pmatrix};\ 0,\ \mu\right) e(-N^{-1} x_2)\, dx_2\, \sigma(w^{-1} \kappa_z w \sigma(J)) \right\}$$

$$\times\, y_1^{-u-3/2} y_2^3 \sqrt{1 + iz}\, dz \frac{dy_1}{y_1}.$$

The expression in braces equals $T\mathscr{C}_0(s; 0, N^{-1}, r; y_1 y_2^{-2} \Delta_z^{-2}, y_2 \Delta_z)$, in the notation of Section 6, and so by Proposition 6.2, $I_6 = I_7 + I_8$, where

$$I_7 = N^{7-s+k/2}\, \hat{\mathscr{L}}(s, 0, r) \int_1^{\infty} \int_{-\infty}^{\infty} T(W^0(y_1 y_2^{-2} \Delta_z^{-2}, N^{-1} y_2 \Delta_z; s)\, \sigma(w\kappa_z J))$$

$$y_1^{-u-3/2} y_2^3 \sqrt{1 + iz}\, dz \frac{dy_1}{y_1},$$

and (since $a((1)=1)$, I_8 equals

$$N^{-s}\int_1^\infty\int_{-\infty}^\infty T(\mathbf{v}\sigma(\eta w^{-1}\kappa_z wJ))\,e^{-2\pi N^{-1}\Delta_z^{-1}y_2}\,\Delta_z^{-s+k/2}\,y_1^{s-u-3/2}\,y_2^{3-s+k/2}\sqrt{1+iz}\,dx\frac{dy_1}{y_1}.$$

Let us consider I_7. Using (3.54), this equals

$$N^{7-s-k/2}\,\hat{\mathscr{L}}(s,0,r)\int_1^\infty\int_{-\infty}^\infty T(W^0(1,N^{-1}y_2\Delta_z;s)\,\sigma(w\kappa_z J))$$

$$y_1^{5/2-s-u}\,y_2^{2s-5}\,\Delta_z^{2s-8}\sqrt{1+iz}\,dz\frac{dy_1}{y_1}=(u+s-5/2)^{-1}\,N^{7-s-k/2}\,\hat{\mathscr{L}}(s,0,r)\,y_2^{2s-5}$$

$$\int_{-\infty}^\infty T(W^0(1,N^{-1}y_2\Delta_z;s)\,\sigma(w\kappa_z J))\,\Delta_z^{2s-8}\sqrt{1+iz}\,dz\,.$$

On the other hand, I_s equals

$$N^{-s}\int_1^\infty\int_{-\infty}^\infty y_1^{s-u-3/2}\,y_2^{3-s+k/2}\,\Delta_z^{-s+k/2}\,e^{-2\pi N^{-1}y_2\Delta_z}\,T(\mathbf{v}\sigma(\eta w^{-1}\kappa_z wJ))$$

$$\sqrt{1+iz}\,dz\frac{dy_1}{y_1}=(u-s+3/2)^{-1}\,N^{-s}\,y_2^{3-s+k/2}$$

$$\int_{-\infty}^\infty \Delta_z^{-s+k/2}\,e^{-2\pi N^{-1}y_2\Delta_z}\,T(\mathbf{v}\sigma(\eta w^{-1}\kappa_z wJ))\sqrt{1+iz}\,dz\,.$$

We have proved that (8.1) equals $I_1-I_4+I_5+I_7+I_8$. Here I_1 and I_5 are holomorphic, while $-I_4$, I_7 and I_8 are the three terms in (8.2). This completes the proof of Proposition 8.1. □

9. Proof of the theorem

In this section, we will prove the Theorem which was stated in the introduction. As noted there, we may assume that the sign in the functional equation of f is $+1$.

Thus far, all we have required of m and N are that $8|N$, $N|m$ and that $4m|N^2$. Now let us further specify that $m=NN_0$, where N_0 is the product of all primes dividing N. Moreover, the integer r which appeared in Sections 6 and 8 we will take to equal 1.

We will analyze the poles of the Dirichlet series $Z^\pm(u,s)$ near $s=2$. By Proposition 8.1, the poles corresponding to the three terms in (8.2) are at $u=s-5/2$, $5/2-s$ and $s-3/2$. When $s=2$, the first pole is farther to the left, and will not play any role in our considerations. The second two poles must be looked at very closely because when $s=2$ they coalesce—it must be shown that their contributions do not cancel.

Moreover, we have the problem of separating the contributions of the positive and negative discriminants. To accomplish this, we use two different representations $\sigma_1\colon K\to GL(\mathbf{V}_1)$ and $\sigma_2\colon K\to GL(\mathbf{V}_2)$, with vectors $\mathbf{v}_i\in\mathbf{V}_i$ satisfying (1.12) and linear functionals T_i on $\mathbf{V}_i$. There are also to be chosen two different values of y_2, which we will denote $y_2^{(1)}$ and $y_2^{(2)}$. We are assuming that Proposition 3.11 is valid for the particular choices of $\mathbf{v}_i$ and T_i, so that Proposition 8.1 is applicable.

We will denote

$$(4m)^{-s+u+5/2} N^{-s+4+k/2} T_i \mathscr{F}^{\pm}(u, s, y_2^{(i)}; \sigma_i) = \mathscr{F}_i^{\pm}(u, s) , \tag{9.1}$$

$$N^{7-s-k/2} y_2^{2s-5} T_i \hat{\mathscr{M}}(s, N^{-1} y_2^{(i)}; \sigma_i) = \mathscr{M}_i(s) , \tag{9.2}$$

and

$$N^{-s} y_2^{3-s+k/2} T_i \tau(s, N^{-1} y_2^{(i)}; \sigma_i) = \tau_i(s) . \tag{9.3}$$

By Proposition 3.12, we may assume that $\mathscr{F}_1^+(1/2, 2) \neq 0$, and that $\mathscr{F}^-(u, s)$ has analytic continuation to a neighborhood of $(u, s) = (1/2, 2)$, but that $\tau_1(2) = 0$. Furthermore, by Proposition 3.13, we may assume that $\tau_2(2) \neq 0$, but make no assumption on $\mathscr{F}_2^{\pm}(u, s)$ except analytic continuation to a neighborhood of $u = 1/2$, $s = 2$. We will make the following notation: If f_1 and f_2 are functions of u and s, we will use the notation $f_1 \sim f_2$ to indicate that $f_1 - f_2$ is a holomorphic function of u and s in a neighborhood of the point $(u, s) = (1/2, 2)$.

Thus we have

$$\mathscr{F}_1^+(u, s)Z^+(u, s) + \mathscr{F}_1^-(u, s)Z^-(u, s) \sim (u + s - 5/2)^{-1} \hat{\mathscr{L}}(s, 0, 1)\mathscr{M}_1(s) \tag{9.4}$$

and

$$\mathscr{F}_2^+(u,s)Z^+(u, s) + \mathscr{F}_2^-(u,s)Z^-(u, s)$$
$$\sim (u + s - 5/2)^{-1} \hat{\mathscr{L}}(s, 0, 1)\mathscr{M}_2(s) + (u - s + 3/2)^{-1} \tau_2(s) . \tag{9.5}$$

Now multiplying (9.4) by $\mathscr{F}_2^+(u,s)$, and (9.5) by $\mathscr{F}_1^+(u,s)$ and subtracting, we obtain

$$\mathscr{F}(u,s)Z^-(u, s) \sim (u + s - 5/2)^{-1} p(u, s) - (u - s + 3/2)^{-1} q(u, s), \tag{9.6}$$

where we denote

$$\mathscr{F}(u,s) = \mathscr{F}_2^+(u,s)\mathscr{F}_1^-(u,s) - \mathscr{F}_1^+(u,s)\mathscr{F}_2^-(u,s) ,$$
$$p(u, s) = [\mathscr{F}_2^+(u,s)\mathscr{M}_1(s) - \mathscr{F}_1^+(u,s)\mathscr{M}_2(s)]\hat{\mathscr{L}}(s, 0, 1) ,$$
$$q(u, s) = \mathscr{F}_1^+(u,s)\tau_2(s) .$$

Let us point out that if D occurs with a nonzero coefficient in $Z^{\pm}(u, s)$, then since $D \equiv 1 \bmod 4m/N$, we have $D = D_0 D_1^2$, where D_0 is a fundamental discriminant, and D_0 is congruent to a square modulo $4N_0$. It follows that $\chi_{D_0}(p) = 1$ for every prime dividing N. Thus every prime dividing N splits completely in $\mathbf{Q}(\sqrt{D})$. Moreover, since $\chi_D(N) = 1$, it follows from (1.24) that the sign in the functional equation of $L(s, f, \chi_D)$ is $+1$ if $D > 0$, and -1 if $D < 0$.

Lemma. *If $D = D_0 D_1^2$, where D_0 is a fundamental discriminant, and if $L(k/2, f, \chi_{D_0}) = 0$, then $\mathscr{L}(2, D) = 0$. If furthermore $L'(k/2, f, \chi_{D_0}) = 0$, then $\mathscr{L}'(2, D) = 0$.*

Proof. The point is to show that $\mathscr{L}(s, D) = b(s) L(s + k/2 - 2, f, \chi D_0)$, where $b(s)$ is holomorphic. If this is known, the first assertion is obvious, and the second follows from the first after using the product rule for derivatives.

Indeed, each factor $\mathscr{L}(s, D)$ is $L_N(2s + k - 4, f, \vee^2)^{-1} L(s + k/2 - 2, f, \chi_{D_0})$ times a finite Dirichlet polynomial and some factors of the form

$$(1 - \sigma_p^2 p^{4-k-2s})^{-1} (1 - \sigma_p'^2 p^{4-k-2s})^{-1} (1 - p^{3-2s})^{-1}$$

where p runs through the primes dividing D, but not dividing N. The holomorphy of $b(s)$ then follows from Proposition 1.2, and the trivial estimate $|\sigma_p| < p^{k/2}$. □

Now, we will show that the left side of (9.6) vanishes when $s = 2$. It is sufficient, by analytic continuation, to check this if $\mathrm{re}(u)$ is sufficiently large, so that the Dirichlet series is convergent. The point is that each individual term $\mathscr{L}(2, D)$ in the Dirichlet series $Z^-(u, s)$, vanishes. Here $L(k/2, f, \chi_{D_0})$ vanishes because there is a minus sign in the functional equation, since $D_0 < 0$. The vanishing of $\mathscr{L}(s, D)$ then follows by Lemma 9.1. Therefore

$$Z^-(u, 2) = 0 . \tag{9.7}$$

Now if we take $s = 2$ in (9.6) and let $u \to 1/2$, (9.7) implies that

$$p(1/2, 2) = q(1/2, 2) . \tag{9.8}$$

Now differentiating (9.6) with respect to s, we obtain

$$\frac{\partial \mathscr{F}}{\partial s}(u, s) Z^-(u, s) + \mathscr{F}(u, s) \frac{\partial Z^-}{\partial s}(u, s) \sim (u + s - 5/2)^{-1} \frac{\partial p}{\partial s}(u, s) -$$

$$(u + s - 5/2)^{-2}\, p(u, s) - (u - s + 3/2)^{-1} \frac{\partial q}{\partial s}(u, s) - (u - s + 3/2)^{-2}\, q(u, s) .$$

Setting $s = 2$, and using (9.7), we obtain

$$\mathscr{F}(u, 2) \frac{\partial Z^-}{\partial s}(u, 2) \sim (u - 1/2)^{-1}$$

$$\left[\frac{\partial p}{\partial s}(u, 2) - \frac{\partial q}{\partial s}(u, 2)\right] - (u - 1/2)^{-2} [p(u, 2) + q(u, 2)] . \tag{9.9}$$

(The meaning of $\sim$ in this equation is that the difference between the left and right hand sides is a holomorphic function of u near $u = 1/2$.) Now $q(1/2, 2) = \mathscr{F}_1^+(1/2, 2)\, \tau_2(2)$. Since we have used Propositions 3.12 and 3.13 to guarantee that $\mathscr{F}_1^+(1/2, 2) \neq 0$ and $\tau_2(2) \neq 0$, we see that $q(1/2, 2) \neq 0$, and so by 9.8, the coefficient in (9.9) of $(u - 1/2)^{-2}$ does not vanish. It follows that $\mathscr{F}(u, 2) \frac{\partial Z^-}{\partial s}(u, 2)$ has a double pole at $u = 1/2$.

Now since

$$Z^-(u, s) = \sum_{\substack{D < 0 \\ D \equiv r^2 \bmod 4m/N}} \mathscr{L}(s, D)\, |D|^{-u+s-5/2} ,$$

and since we have shown that $\mathscr{L}(2, D) = 0$ for each term in this sum, it follows from the product rule for derivatives that

$$\frac{\partial Z^-}{\partial s}(u, 2) = \sum_{\substack{D < 0 \\ D \equiv r^2 \bmod 4m/N}} \mathscr{L}'(2, D)\, |D|^{-u-1/2} .$$

We have established that this Dirichlet series has a double pole at $u = 1/2$. Hence $\mathscr{L}'(s, D) \neq 0$ for infinitely many $D < 0$.

We now observe that if $D = D_0 D_1^2$, with D_0 a fundamental discriminant, and if $\mathscr{L}'(2, D) \neq 0$, then $L'(k/2, f, \chi_{D_0}) \neq 0$. Indeed, this follows from Lemma 9.1. There is one slight subtlety, however: we have proved that infinitely many $\mathscr{L}'(2, D) \neq 0$. We must further show that these infinitely many nonvanishing $\mathscr{L}'(2, D)$ do not all correspond to some finite set of fundamental discriminants. Suppose that this were the case. Then there would be some $D_0 < 0$, congruent to a square modulo $4m/N$, such that the Dirichlet subseries

$$\sum_{D_1} \mathscr{L}'(2, D_0 D_1^2)\,|D_0 D_1^2|^{-u+1/2}$$

had a double pole at $u = 1/2$. However, Proposition 7.1 asserts that this is not the case. We have therefore proved part (i) of our Theorem.

Now let us turn to the proof of part (ii)—Waldspurger's Theorem. We choose a representation $\sigma_3\colon K \to GL(\mathbf{V}_3)$, a vector $\mathbf{v}_3 \in \mathbf{V}_3$, another linear functional T_3 on $\mathbf{V}_3$, and another value $y_2^{(3)}$. Again, we assume that Proposition 3.11 is applicable. We make the definitions (9.1–3) again. This time, we use Proposition 3.15 to assume that $\mathscr{M}_3(2) \neq 0$, but $\tau_3(2) = 0$. We have

$$\mathscr{F}_3^+(u,s)Z^+(u,s) + \mathscr{F}_3^-(u,s)Z^-(u,s) \sim (u+s-5/2)^{-1}\hat{\mathscr{L}}(s,0,1)\mathscr{M}_3(s). \quad (9.10)$$

Now multiplying (9.2) by $\mathscr{M}_3(2)$, and multiplying (9.3) by $\mathscr{M}_2(2)$ and subtracting, we obtain

$$[\mathscr{M}_3(2)\mathscr{F}_2^+(u,s) - \mathscr{M}_2(2)\mathscr{F}_3^+(u,s)]Z^+(u,s) + [\mathscr{M}_3(2)\mathscr{F}_2^-(u,s) - \mathscr{M}_2(2)\mathscr{F}_3^-(u,s)]Z^-(u,s) \sim (u+s-5/2)^{-1}[\mathscr{M}_3(2)\mathscr{M}_2(s) - \mathscr{M}_2(2)\mathscr{M}_3(s)] + (u-s+3/2)^{-1}\mathscr{M}_3(s)\tau_2(s)\,.$$

Now substituting $s = 2$ and recalling that $Z^-(u,2) = 0$, we obtain

$$[\mathscr{M}_3(2)\mathscr{F}_2^+(u,2) - \mathscr{M}_2(2)\mathscr{F}_3^+(u,2)]Z^+(u,2) \sim (u-1/2)^{-1}\mathscr{M}_3(2)\tau_2(2)\,.$$

Since $\mathscr{M}_3(2)\tau_2(2) \neq 0$, we see that $Z^+(u,2)$ has a pole. Hence infinitely many $\mathscr{L}(s,D)$ do not vanish with $D > 0$. The remainder of the proof is the same as the proof of part (i). This completes the proof of the Theorem.

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