Computing Bernoulli Numbers Quickly

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The Bernoulli numbers are defined via the coefficients of the power series expansion of $t/(e^t - 1)$. Namely, for integers $m \geq 0$ we define B_m so that

$$\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} \frac{B_m}{m!} t^m.$$

Multiplying both sides above by $e^t - 1$ and equating coefficients of t^{m+1} yields:

$$B_0 = 1, \qquad (m+1)B_m = -\sum_{k=0}^{m-1} \binom{m+1}{k} B_k$$

Some authors take the above recurrence to be the definition of the Bernoulli numbers. This recurrence provides a straightforward method for calculating B_m and is especially convenient for computing B_m for all m up to some bound. The first few Bernoulli numbers are:

$$B_0 = 1, \quad B_1 = -\frac{1}{2} \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30},$$

$$B_5 = 0, \quad B_6 = \frac{1}{42}, \quad B_7 = 0, \quad B_8 = -\frac{1}{30}, \quad B_9 = 0,$$

$$B_{10} = \frac{55}{66}\ , \ B_{11} = 0, \quad B_{12} = -\frac{691}{2730} \quad B_{13} = 0, \quad B_{14} = -\frac{7}{6}$$

The values above provide evidence for two basic results regarding Bernoulli numbers. First, $B_m = 0$ for odd $m \geq 3$, and secondly, for even $m \geq 2$, $B_m = (-1)^{m/2+1} |B_m|$. Henceforth we will denote

$$B_m = \frac{a}{d}$$

where $a, d \in \mathbb{Z}$, $d \geq 1$, and $(a, d) = 1$. From the properties mentioned above, it is clear that $a = (-1)^{m/2+1} |a|$ for even $m \geq 2$. The goal of this paper is to compute B_m rapidly, when m is potentially very large. Computing B_m via the recurrence is slow; it requires us to sum over $m(m+1)/2$ terms. In addition, this method requires storing the numbers $B_0, \ldots, B_{m-1}$ in memory. In order to speed up this computation, we will describe an important connection between the Bernoulli numbers and the Riemann Zeta Function. Much of what we will describe was gleaned from the PARI-2.2.11.alpha source code. The algorithm this version of PARI uses to compute Bernoulli numbers was written by Henri Cohen and later refined by Karim Belabas; it was originally designed to speed up the computation of zeta values. For real $s > 1$, Euler defined the function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s},$$

for which he proved the product formula

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}.$$

(Riemann showed that $\zeta(s)$ has an analytic continuation to the entire complex plane with a simple pole at $s = 1$, and hence the function bears his name.) In addition to giving the product formula, Euler was able to evaluate the zeta function at the even positive integers. [1] For any integer $m \geq 1$,

$$2\zeta(2m) = \frac{(-1)^{m+1}(2\pi)^{2m}}{(2m)!} B_{2m}.$$

It follows that for any even integer $m \geq 4$,

$$|B_m| = \frac{2m!}{(2\pi)^m} \zeta(m).$$

It is now clear how to compute decimal approximations to B_m; we merely approximate $\zeta(m)$ using the Euler product and plug the result into the above equation. A priori, this is not enough to compute B_m as a ratio of integers, but fortunately a theorem of Clausen and von Staudt precisely describes the denominator of B_m in terms of the divisors of m. [1] For even $m \geq 2$,

$$d := \operatorname{denom}(B_m) = \prod_{p-1|m} p.$$

Now we show how to compute a. First define

$$K := \frac{2m!}{(2\pi)^m}$$

so that $|B_m| = K\zeta(m)$. Using the Euler product, we may approximate $\zeta(m)$ from below with arbitrary precision. Suppose that we have computed a number z such that

$$0 \leq \zeta(m) - z < (Kd)^{-1}\,;$$

then we have

$$0 \leq |B_m| - zK < d^{-1}$$

and hence

$$0 \leq |a| - zKd < 1\,.$$

It follows that $|a| = \lceil zKd \rceil$ and hence $a = (-1)^{m/2+1} \lceil zKd \rceil$. It remains to explicitly compute z. In order to accomplish this, we consider the following problem: given $s > 1$ and $\varepsilon > 0$, find $N \in \mathbb{Z}^+$ so that when we set

$$z := \prod_{p \leq N} (1 - p^{-s})^{-1}\,,$$

we are guaranteed that $0 \leq \zeta(s) - z < \varepsilon$. We always have $0 \leq \zeta(s) - z$. Further, it is not hard to see that

$$\sum_{n \leq N} n^{-s} \leq \prod_{p \leq N} (1 - p^{-s})^{-1}$$

and therefore

$$\begin{aligned} \zeta(s) - z &\leq \sum_{n=N+1}^{\infty} n^{-s} \\ &\leq \int_N^{\infty} x^{-s}\,dx \\ &= \frac{1}{(s-1)N^{s-1}}\,. \end{aligned}$$

If we choose $N > \varepsilon^{-1/(s-1)}$, then we have

$$\frac{1}{(s-1)N^{s-1}} \leq \frac{1}{N^{s-1}} < \varepsilon\,,$$

which implies $\zeta(s) - z < \varepsilon$, as required. For our purposes, we have $s = m$ and $\varepsilon = (Kd)^{-1}$ and therefore it suffices to choose $N > (Kd)^{1/(m-1)}$.

The Algorithm: Suppose $m \geq 2$ is even.

1. $$K := \frac{2m!}{(2\pi)^m}$$

2. $$d := \prod_{p-1|m} p$$

3. $$N := \left\lceil (Kd)^{1/(m-1)} \right\rceil$$

4. $$z := \prod_{p \leq N} (1 - p^{-m})^{-1}$$

5. $$a := (-1)^{m/2+1} \lceil dKz \rceil$$

6. $$B_m = \frac{a}{d}$$

Some remarks are in order. In step (1), we must be careful to compute K to sufficient precision so that the calculation in (5) gives the desired result. In order to compute (4), it is useful to first compute all primes $p \leq N$; this may be done quickly using the Sieve of Eratosthenes. One may also compute the product in (2) via a sieving process. Finally, for the value of N we may choose any integer greater than or equal to the one specified in (3), so we need not worry about computing $(Kd)^{1/(m-1)}$ to much precision. It is interesting to note that the algorithm above also gives a way of approximating $\zeta(m)$ quickly for even m. Namely, compute B_m as a rational number using this algorithm and plug it into Euler's formula for $\zeta(m)$ along with an approximation of π sufficiently many decimal places.

Example: We use the modest size example of $m = 50$ for the sake of readability. Using 50 digits of precision, we compute

$$K = 750086674607695770474773 6.71552473164564038043676 04\,.$$

The divisors of m are $1, 2, 5, 10, 25, 50$ and hence

$$d = (2)(3)(11) = 66\,.$$

We find $N = 4$ and compute

$$z = 1.0000000000000008881784210930815902983501390146827\,.$$

Finally we compute

$$dKz = 49505720524107964821247752 4.9999999944261511121065 2$$

and therefore

$$B_{50} = \frac{495057205241079648212477525}{66}\,.$$

References

[1] Ireland, Kenneth. Rosen, Michael. *A Classical Introduction to Modern Number Theory.* New York: Springer-Verlag, 1990.