3.4 Computing a basis for $S_2(\Gamma_0(N))$

In this section we explain a method for using what we know how to compute using modular symbols to compute a basis for $S_2(\Gamma_0(N))$.

Let $\mathbb{M}_2(\Gamma_0(N);\mathbb{Q})$ and $\mathbb{S}_2(\Gamma_0(N);\mathbb{Q})$ denote modular symbols and cuspidal modular symbols over $\mathbb{Q}$. Before we begin, we describe a simple but crucial fact about the relation between cusp forms and the Hecke algebra.

If $f = \sum b_n q^n \in \mathbb{C}[[q]]$ is a power series, let $a_n(f) = b_n$ be the n coefficient of f. Notice that a_n is a linear map from $\mathbb{C}[[q]]$ to itself.

As explained in [Lan95, §VII.3], the Hecke operators T_n acts on elements of $M_2(\Gamma_0(N))$ as follows:

$$T_n\left(\sum_{m=0}^{\infty} a_m q^m\right) = \left(\sum_{1\leq d \mid \gcd(n,m)} \varepsilon(d)\cdot d\cdot a_{mn/d^2}\right) q^m, \tag{3.4.1}$$

where $\varepsilon(d) = 1$ if $\gcd(d,N) = 1$ and $\varepsilon(d) = 0$ if $\gcd(d,N) \neq 1$.

Lemma 3.4.1. *Suppose f is a modular form and n is a positive integer. Then*

$$a_1(T_n(f)) = a_n(f).$$

Proof. The coefficient of q in (3.4.1) is $\varepsilon(1)\cdot 1\cdot a_{1\cdot n/1^2} = a_n$. □

Let $\mathbb{T}'$ denote the image of the Hecke algebra in $\mathrm{End}(S_2(\Gamma_0(N)))$, and let $\mathbb{T}'_{\mathbb{C}} = \mathbb{T}' \otimes \mathbb{C}$ be the $\mathbb{C}$-span of the Hecke operators.

Proposition 3.4.2. *There is a perfect bilinear pairing of complex vector spaces*

$$S_2(\Gamma_0(N)) \times \mathbb{T}'_{\mathbb{C}} \to \mathbb{C}$$

given by

$$\langle f, t\rangle = a_1(t(f)).$$

Proof. The pairing is bilinear since both t and a_1 are linear. Suppose $f \in S_2(\Gamma_0(N))$ is such that $\langle f, t\rangle = 0$ for all $t \in \mathbb{T}'_{\mathbb{C}}$. Then in particular $\langle f, T_n\rangle = 0$ for each positive integer n. But by Lemma 3.4.1 we have

$$a_n(f) = a_1(T_n(f)) = 0$$

for all n; thus $f = 0$.

Next suppose that $t \in \mathbb{T}'_{\mathbb{C}}$ is such that $\langle f, t\rangle = 0$ for all $f \in S_2(\Gamma_0(N))$. Then $a_1(t(f)) = 0$ for all f. For any n, the image $T_n(f)$ is also a cuspform, so $a_1(t(T_n(f))) = 0$ for all n and f. Finally $\mathbb{T}'$ is commutative and Lemma 3.4.1 together imply that for all n and f,

$$0 = a_1(t(T_n(f))) = a_1(T_n(t(f))) = a_n(t(f)),$$

so $t(f) = 0$ for all f. Thus t is the 0 operator. □

By Proposition 3.4.2 there is an isomorphism of vector spaces

$$\Psi : S_2(\Gamma_0(N)) \xrightarrow{\cong} \mathrm{Hom}(\mathbb{T}', \mathbb{C})$$

that sends $f \in S_2(\Gamma_0(N))$ to the homomorphism

$$t \mapsto a_1(t(f)).$$

For any linear map $\varphi : \mathbb{T}'_{\mathbb{C}} \to \mathbb{C}$, let

$$f_\varphi = \sum_{n=1}^{\infty} \varphi(T_n) q^n \in \mathbb{C}[[q]].$$

By Lemma 3.4.1, we have

$$\langle f_\varphi, T_n \rangle = a_1(T_n(f_\varphi)) = a_n(f_\varphi) = \varphi(T_n).$$

Thus f_φ must be the q-expansion of the modular form that corresponds to φ under the isomorphism Ψ. In paritcular, $f_\varphi \in S_2(\Gamma_0(N))$, and the cuspforms f_φ, as φ runs through a basis, form a basis for $S_2(\Gamma_0(N))$.

> *We can compute $S_2(\Gamma_0(N))$ by computing $\mathrm{Hom}(\mathbb{T}', \mathbb{C})$, where we compute $\mathbb{T}'$ in any way we want, e.g., using a space that contains an isomorphic copy of $S_2(\Gamma_0(N))$.*

Algorithm 3.4.3 (Basis of Cuspforms)**.** *Given a positive integers N and B, this algorithm computes a basis for $S_2(\Gamma_0(N))$ to precision $O(q^B)$.*

1. Compute the modular symbols space $\mathbb{M}_2(\Gamma_0(N); \mathbb{Q})$ via the presentation of Section 3.2.2.
2. Compute the subspace $\mathbb{S}_2(\Gamma_0(N); \mathbb{Q})$ of cuspidal modular symbols as in Section 3.3.
3. Let $d = \frac{1}{2} \cdot \dim \mathbb{S}_2(\Gamma_0(N); \mathbb{Q})$. This is the dimension of $S_2(\Gamma_0(N))$.
4. Use the Hecke operators T_2, T_3, etc., of Section 3.2.3 to find the unique subspace V of $\mathrm{Hom}(\mathbb{M}_2(\Gamma_0(N); \mathbb{Q}), \mathbb{Q})$ that is isomorphic to $\mathbb{S}_2(\Gamma_0(N); \mathbb{Q})$ as a $\mathbb{T}$-module. (The Hecke operators act via their transpose; find the subspace V of the dual with the same characteristic polynomials.)
5. Let $[T_n]$ denote the matrix of T_n acting on some fixed basis of V. For a matrix A, let $a_{ij}(A)$ denote the ij-th entry of A. For various integers i, j with $0 \le i, j \le d-1$, compute formal q-expansions

$$f_{ij}(q) = \sum_{n=1}^{B-1} a_{ij}([T_n]) q^n + O(q^B) \in \mathbb{Q}[[q]]$$

   until we find enough to span a space of dimension d (or exhaust all of them, in which case B is too small). These f_{ij} then form a basis for $S_2(\Gamma_0(N))$.

3.4.1 Examples

In this section we use SAGE to demonstrate Algorithm 3.4.3 for computing $S_2(\Gamma_0(N))$ for various N.

Example 3.4.4. The smallest N with $S_2(\Gamma_0(N)) \neq 0$ is $N = 11$.

```
sage: M = ModularSymbols(11)
sage: M.basis()
((1,0), (1,8), (1,9))
sage: S = M.cuspidal_subspace()
sage: S
Dimension 2 subspace of a modular symbols space of level 11
sage: S.basis()
((1,8), (1,9))
sage: d = S.dimension() // 2; d
1
```

The command `dual_free_module` computes the vector space V of Algorithm 3.4.3.

```
sage: S.dual_free_module()
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[1 0 5]
[0 1 0]
```

View each of the basis vectors $(1,0,5)$ and $(0,1,0)$ as defining a linear map (via dot product) $\mathbb{S}_2(\Gamma_0(11)) \to \mathbb{Q}$, where we view elements of $\mathbb{S}_2(\Gamma_0(11))$ as linear combinations of our fixed basis $(1,0)$, $(1,8)$, $(1,9)$ for $\mathbb{M}_2(\Gamma_0(11))$.

The command `dual_hecke_matrix` computes the matrix of T_n on the above basis for V.

```
sage: S.dual_hecke_matrix(1)
[1 0]
[0 1]
sage: S.dual_hecke_matrix(2)
[-2  0]
[ 0 -2]
sage: S.dual_hecke_matrix(3)
[-1  0]
[ 0 -1]
```

Thus

$$f_{0,0} = q - 2q^2 - q^3 + \cdots \in S_2(\Gamma_0(11)).$$

Since $\dim S_2(\Gamma_0(11)) = 1$, this form must be a basis.

Example 3.4.5. Next consider $N = 23$, where we have $d = \dim S_2(\Gamma_0(23)) = 2$. The command `q_expansion_cuspforms` computes V and the matrices $[T_n]|V$

and returns a function f such that $f(i,j)$ is the q-expansion of $f_{i,j}$ to some precision.

```
sage: M = ModularSymbols(23)
sage: S = M.cuspidal_subspace()
sage: S
Dimension 4 subspace of a modular symbols space of level 23
sage: f = S.q_expansion_cuspforms(6)
sage: f(0,0)
q - 2/3*q^2 + 1/3*q^3 - 1/3*q^4 - 4/3*q^5 + O(q^6)
sage: f(0,1)
O(q^6)
sage: f(1,0)
-1/3*q^2 + 2/3*q^3 + 1/3*q^4 - 2/3*q^5 + O(q^6)
```

Thus a basis for $S_2(\Gamma_0(23))$ is

$$f_{0,0} = q - \frac{2}{3}q^2 + \frac{1}{3}q^3 - \frac{1}{3}q^4 - \frac{4}{3}q^5 + \cdots$$
$$f_{1,0} = -\frac{1}{3}q^2 + \frac{2}{3}q^3 + \frac{1}{3}q^4 - \frac{2}{3}q^5 + \cdots$$

Or, in echelon form,

$$q - q^3 - q^4 + \cdots$$
$$q^2 - 2q^3 - q^4 + 2q^5 + \cdots$$

which we computed using

```
sage: S.q_expansion_basis(6)
 [q - q^3 - q^4 + O(q^6),
    q^2 - 2*q^3 - q^4 + 2*q^5 + O(q^6)]
```