Mathematics 21b.
Linear Algebra and Differential Equations

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1 Introduction

According to **Fact 2.2.3** of the textbook, a linear transformation $T : \mathbb{R}^m \longrightarrow R^n$ can only be invertible if $m = n$ (note that this is a necessary, but not sufficient condition). The purpose of this handout is to provide you with an example of an invertible, non-linear transformation "from $\mathbb{R}$ to $\mathbb{R}^2$." In actuality, the transformation I will give is really only from an open subset of $\mathbb{R}$ to an open subset of $\mathbb{R}^2$, but the sheer fact that I can construct such a transformation is nonetheless impressive, simply because it is counter-intuitive. Indeed, one might prematurely think that the creation of such an invertible (*i.e.,* one-to-one) transformation is impossible, since there seem to be many more points in any open subset of $\mathbb{R}^2$ than there are in any open subset of $\mathbb{R}$. Here is where the concept of infinity makes things tricky!

2 Notations and Conventions

- Let $S = \{x \in \mathbb{R} \ : \ 0 < x < 1\}$. Hence, S is the open interval from 0 to 1 on the real number line. Let $T = \{(x, y) \in \mathbb{R}^2 \ : \ 0 < x < 1, 0 < y < 1\}$. Hence, T is the "open unit square" which fits snugly in the corner of the first quadrant of the Euclidean plane.
- From now on, every terminating decimal number will be re-written with an infinite string of nines at the end. For example, $0.56 = 0.59999999\ldots$. By doing this, we eliminate any and all ambiguity about the representation of real numbers as decimals.

3 Mapping the Unit Interval to the Unit Square

Now for the heart of the matter: we are going to create a one-to-one, invertible (note that the two are equivalent) transformation from the unit interval to the unit square. Before we proceed, think for a moment about the implications of the existence of such a map. Perhaps most shockingly: *The number of points* *in the unit interval is equal to the number of points in the unit square!* Such a statement, from a geometric point of view, is completely remarkable! By the way, the technical mathematical definition for this equality of set sizes is called **equipotency**. Hence, S is equipotent with T, and we write $\#S = \#T$, or $|S| = |T|$.

Theorem 1. *There exists an invertible map $f : S \longrightarrow T$.*

Proof. Let $x \in S$. Then, we can represent x as a decimal number in a unique way (see Section 2), say, $x = 0.a_0a_1a_2a_3a_4a_5\ldots$, where $a_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Consider the point $y = (0.a_0a_2a_4\ldots, 0.a_1a_3a_5\ldots)$. This is clearly in T. Hence, every real number in S corresponds to a unique point in T. Similarly, we can take any point in T and construct a unique number in S in exactly the same way. Hence, the map $f(0.a_0a_1a_2a_3a_4a_5\ldots) = (0.a_0a_2a_4\ldots, 0.a_1a_3a_5\ldots)$ is invertible, as desired. $\square$

4 Exercises

1. Show that $|\mathbb{Z}| = |\mathbb{N}|$. Here, $\mathbb{Z}$ is the set of integers, and $\mathbb{N}$ is the set of natural numbers (including 0).

2. Is $|\mathbb{Z}| = |\mathbb{Q}|$? Here, $\mathbb{Q}$ is the set of rational numbers.

3. Formulate a conjecture as to whether or not $|\mathbb{R}| = |\mathbb{N}|$. Then, use the web to find out what is actually the case!