Math 581e, Fall 2012, Homework 8

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Due: Friday, November 30, 2012

This is the last homework assignment. There are 3 problems. Turn your solutions in Friday, November 30, 2012 in class. You may work with other people and can find the LaTeX of this file at http://wstein.org/edu/2012/ant/hw/. If you use Sage to solve a problem, include your code in your solution. I have office hours 12:30–2:00 on Wednesdays in Padelford C423.

1. Turn in rough draft of your project.

2. Let K be the cubic field obtained by adjoining a root of $f =$ to $\mathbb{Q}$.

   (a) Show that $r = 3, s = 0$, i.e., there are 3 real embeddings.

   (b) Compute (e.g., using Sage) explicit generators for the unit group U_K.

   (c) Draw a picture that illustrates how U_K maps to a lattice in a codimension one subspace of $\mathbb{R}^3$.

   (d) Choose a basis for the image of U_K, and compute the 2×2 matrix A corresponding to the dot product pairing on that basis.

   (e) Compute the absolute value of the determinant of A, which is (basically) a quantity called the *regulator* of the number field K.

3. (This problem is inspired by Aly's talk about pseudo-basis, and is Lemma 1.2.20 of `http://wstein.org/5canz/craig/math/Cohen%20--%20Advanced%20topics%20in%20computational%20number%20theory.pdf`) Let $\mathfrak{a}$ and $\mathfrak{b}$ be nonzero ideals of a Dedekind domain R. Prove that the R-modules $\mathfrak{a} \oplus \mathfrak{b}$ and $R \oplus \mathfrak{a}\mathfrak{b}$ are isomorphic, as follows:

   (a) Reduce to the case that $\mathfrak{a}$ and $\mathfrak{b}$ are integral ideals, by using that $\mathfrak{a} \cong \mathfrak{a}(\alpha)$ for nonzero α.

   (b) Use the lemma we proved "there is α such that $\alpha I^{-1} \subset R$ is coprime to J" to reduce to the case when $\mathfrak{a}$ and $\mathfrak{b}$ are coprime.

   (c) Define a map $f : \mathfrak{a} \oplus \mathfrak{b} \to R$ by $f(a, b) = a - b$. Show that f is an R-module homomorphism, then use that $\mathfrak{a}$ and $\mathfrak{b}$ are coprime to deduce that we have an exact sequence
   $$0 \to \mathfrak{a} \cap \mathfrak{b} \to \mathfrak{a} \oplus \mathfrak{b} \to R \to 0.$$

   (d) Understand this: Since R is a free module, it is projective, which implies that the above exact sequence splits, which proves the statement.