# Math 581e, Fall 2012, Homework 3

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Due: Friday, October 19, 2012

There are 4 problems. Turn your solutions in Friday, October 19, 2012 in class. You may work with other people and can find the LaTeX of this file at http://wstein.org/edu/2012/ant/hw/. If you use Sage to solve a problem, include your code in your solution. I have office hours 12:30–2:00 on Wednesdays in Padelford C423.

1. Consider the ring $R = \mathbb{Z}[\sqrt{-5}]$.
   (a) Prove that $R$ is a Dedekind domain.
   (b) Prove that the number 6 factors in two completely different ways as a product of irreducible elements. [*Irreducible* means "not a product of two non-units".]
   (c) Explicitly factor $6R$ as a product of prime ideals. [You can use Sage for this.]
2. Let $\overline{\mathbb{Z}}$ be the ring of all algebraic integers in a fixed choice of $\overline{\mathbb{Q}}$. Then $\overline{\mathbb{Z}}$ is *not* a Dedekind domain. Which of the three properties of a Dedekind domain does the ring $\overline{\mathbb{Z}}$ satisfy (give proof)? [The properties are: (1) every nonzero prime is maximal, (2) noetherian, (3) integrally closed in its field of fractions.]
3. Use Sage to find a basis for the ring of integers of the number field obtained by adjoining one root of the polynomial $x^3 + x^2 - 2x + 8$ to $\mathbb{Q}$.
4. Let $K$ be a field of the form $\mathbb{F}_q(t)[x]/(g)$ with $\mathbb{F}_q$ a finite field and $g \neq 0$, so $K$ is obtained from $\mathbb{F}_q(t)$ by adjoining one algebraic element. Let $\mathcal{O}_K$ be the ring of elements of $K$ that satisfy a nonzero monic polynomial with coefficients in $\mathbb{F}_q[t]$. Make the further assumption that $\mathcal{O}_K$ is finitely generated as an $\mathbb{F}_q[t]$ module. Prove that every prime ideal of $\mathcal{O}_K$ is maximal.