Math 581e, Fall 2012, Homework 1

William Stein (wstein@uw.edu)

Due: Friday, October 5, 2012

There are 3 problems. Turn your solutions in Friday, October 5, 2012 in class. You may work with other people and can find the LaTeX of this file at http://wstein.org/edu/2012/ant/hw/. Reminder: I have office hours 1:00pm-2:30pm on Wednesdays in Padelford C423.

1. Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial, and let α be a root of f. Prove that the set of polynomial expressions in α of degree at most $\deg(f) - 1$ is a field. As I explained in class, the only thing that is not obvious is that every nonzero element has an inverse, so that is the only thing you need to prove in giving your solution to this problem. [Consider a nonzero $\beta = \sum_{i=0}^{\deg(f)-1} a_i \alpha^i$ and let $g(x) = \sum a_i x^i$, then use the extended Euclidean algorithm to show there exists $h(x), m(x) \in \mathbb{Q}[x]$ with $h(x)f(x) + m(x)g(x) = 1$.]

2. Prove that the following additive abelian groups are not finitely generated:

$$\mathbb{Q},\ \mathbb{Q}/\mathbb{Z},\ \mathbb{R},\ (\mathbb{Z}/7\mathbb{Z})^{\infty},$$

(this last group is a countably-infinite dimensional vector space over the field of cardinality 7). [One trick – if a nonzero group is *divisible*, then it can't be finitely generated.]

3. Let A be the matrix

$$A = \begin{pmatrix} 0 & 7 & 1 \\ 0 & -1 & 1 \\ -12 & 0 & 4 \end{pmatrix}.$$

Compute the Smith normal form of A and the cokernel of the map $\mathbb{Z}^3 \to \mathbb{Z}^3$ defined by $v \mapsto Av$. Do the same for the matrix

$$B = \begin{pmatrix} 0 & 6 \\ -1 & 7 \\ -1 & -3 \end{pmatrix}$$