Some definitions of Hecke operators

Math 581G Project

Author:
Bharathwaj Palvannan

Instructor:
William Stein

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1. The double coset operator

Note: The following section is taken from [1]

For each $\alpha \in GL_2^+(\mathbb{Q})$, we call the set

$$\Gamma_1\alpha\Gamma_2 = \{\gamma_1\alpha\gamma_2 : \gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2\}.$$

a double coset. The group Γ_1 acts on the double coset $\Gamma_1\alpha\Gamma_2$ through left multiplication and Γ_2 acts through right multiplication. For the rest of the discussion, Γ_1 and Γ_2 will represent congruence subgroups of $SL_2(\mathbb{Z})$.

In this section, we discuss how the double coset $\Gamma_1\alpha\Gamma_2$ can be viewed as a linear map from $M_k(\Gamma_1)$ to $M_k(\Gamma_2)$.

Lemma 1.1. *Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$. Let $\alpha \in GL_2^+(\mathbb{Q})$. Then $\alpha^{-1}\Gamma\alpha \cap SL_2(\mathbb{Z})$ is a congruence subgroup of $SL_2(\mathbb{Z})$.*

Proof. There exists $N_1 \in \mathbb{Z}^+$ satisfying the conditions $\Gamma(N_1) \subset \Gamma$, $N_1\alpha \in M_2(\mathbb{Z})$ and $N_1\alpha^{-1} \in \mathbb{Z}$. Set $N = N_1^3$. We see that

$$\alpha\Gamma(N)\alpha^{-1} \subset \alpha(I + N_1^3 M_2(\mathbb{Z}))\alpha^{-1} \subset I + N_1 M_2(\mathbb{Z})$$

Also $\alpha\Gamma(N)\alpha^{-1}$ consists of matrices with determinant 1. So we can conclude that $\alpha\Gamma(N)\alpha^{-1} \subset \Gamma(N_1)$. Intersecting with $SL_2(\mathbb{Z})$ gives us that $\alpha^{-1}\Gamma\alpha \cap SL_2(\mathbb{Z})$ is a congruence subgroup.

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Lemma 1.2. *Let $\alpha \in GL_2^+(\mathbb{Q})$. Set $\Gamma_3 = \alpha^{-1}\Gamma_1\alpha \cap \Gamma_2$, a subgroup of Γ_2. Then left multiplication by α*

$$\Gamma_2 \to \Gamma_1\alpha\Gamma_2 \qquad \text{given by} \qquad \gamma_2 \to \alpha\gamma_2$$

induces a natural bijection from the coset space $\Gamma_3 \setminus \Gamma_2$ to the orbit space $\Gamma_1 \setminus \Gamma_1\alpha\Gamma_2$. In particular, the orbit space $\Gamma_1 \setminus \Gamma_1\alpha\Gamma_2$ is finite.

Proof.
- The induced map is well defined since if $\gamma_3 = \alpha^{-1}\gamma_1\alpha \in \Gamma_3$, for some $\gamma_1 \in \Gamma_1$ and if $\gamma_2 \in \Gamma_2$, then $\alpha\gamma_3\gamma_2 = \alpha\alpha^{-1}\gamma_1\alpha\gamma_2 = \gamma_1\alpha\gamma_2$, which is equivalent to $\alpha\gamma_2$ in the same orbit space $\Gamma_1 \setminus \Gamma_1\alpha\Gamma_2$.
- The induced map is clearly surjective since for any $\gamma_1 \in \Gamma_1$, $\gamma_2 \in \Gamma_2$, $\gamma_1\alpha\gamma_2$ is equivalent to $\alpha\gamma_2$ in the orbit space $\Gamma_1 \setminus \Gamma_1\alpha\Gamma_2$.
- If γ_2 and $\gamma_2^{'} \in \Gamma_2$ map onto the same element of the orbit space, then there exists $\gamma_1 \in \Gamma_1$ such that $\gamma_1\alpha\gamma_2 = \alpha\gamma_2^{'} \implies \gamma_2^{'}\gamma_2^{-1} = \alpha^{-1}\gamma_1\alpha \in \Gamma_3$. Thus the induced map is injective.

This proves that the induced map is bijective.
By Lemma 1.1, we have that Γ_3 is a congruence subgroup. So we can say that $\Gamma_3 \supset \Gamma(N_3)$ for some natural number N_3. Thus

$$\Gamma(N_3) \subset \Gamma_3 \subset \Gamma_2 \subset SL_2(\mathbb{Z}).$$

Using the surjectivity of the natural map $SL_2(\mathbb{Z}) \longrightarrow SL_2\left(\frac{\mathbb{Z}}{N_3\mathbb{Z}}\right)$, we have that $\Gamma(N_3) \setminus SL_2(\mathbb{Z})$ has finite cardinality and this implies that $\Gamma_3 \setminus \Gamma_2$ (and hence $\Gamma_1 \setminus \Gamma_1\alpha\Gamma_2$) has finite cardinality.

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Definition 1.1. If $\alpha \in GL_2^+(\mathbb{Q})$, the weight k-operator takes functions $f \in M_k(\Gamma_1)$ to functions in $M_k(\Gamma_2)$, namely

$$f[\Gamma_1\alpha\Gamma_2]_k = \sum_j f[\beta_j]_k \in M_k(\Gamma_2),$$

where we write $\Gamma_1\alpha\Gamma_2 = \cup_j \Gamma_1\beta_j$, where we choose β_j to be the orbit representatives and the union is consequently a disjoint union.

- Since the orbit space is finite by Lemma 1.2, the sum $\sum_j f[\beta_j]_k$ is well defined.
- If we write $\Gamma_1\alpha\Gamma_2 = \cup_j \Gamma_1\tau_j$, for some other choice of orbit representatives $\{\tau_j\}$, then for each j, there exists γ_j such that $\gamma_j\beta_j = \tau_j$. So $f[\tau_j]_k = f[\gamma_j\beta_j]_k = (f[\gamma_j]_k)[\beta_j]_k = f[\beta_j]_k$. This proves that the action of $\Gamma_1\alpha\Gamma_2$ does not depend on the choice of the orbit representatives.
- If $\gamma \in \Gamma_2$ and if we write $\Gamma_1\alpha\Gamma_2 = \cup_j \Gamma_1\beta_j$ for some choice of orbit representatives $\{\beta_j\}$, then $\Gamma_1\alpha\Gamma_2 = \Gamma_1\alpha\Gamma_2\gamma = \cup_j \Gamma_1\beta_j\gamma$. Thus $\{\beta_j\gamma\}$ also form a set of orbit representatives. Thus we have

  $$f([\Gamma_1\alpha\Gamma_2]_k)[\gamma]_k = (\sum_j f[\beta_j]_k)[\gamma]_k = \sum_j f[\beta_j\gamma]_k = f[\Gamma_1\alpha\Gamma_2]_k.$$

  Thus to conclude that the action of $\Gamma_1\alpha\Gamma_2$ takes $f \in M_k(\Gamma_1)$ to an element of $M_k(\Gamma_2)$, we need to show holomorphy at the cusps. If $\sigma \in GL_2^+(\mathbb{Q})$, then we can write $\sigma = r\cdot\beta$, where $r \in \mathbb{Q}$ and $\beta \in M_2(\mathbb{Z})$. Using elementary row operations, we can write $\beta = \gamma\tau$, where $\gamma \in SL_2(\mathbb{Z})$ and $\tau =$

  $$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

  $\in M_2(\mathbb{Z})$ is upper triangular. Using the above observation and the fourier expansion $f(z) = \sum_n a_n e^{2\cdot\pi\cdot i\cdot n\cdot z}$ at the cusps, we can obtain a fourier expansion for $f[\sigma]_k(z) = \sum_n C\cdot a_n e^{\frac{2\cdot\pi\cdot i\cdot n\cdot a\cdot z}{c}}$. The above observation also shows that the double coset operator takes cusp forms of Γ_1 to cusp forms of Γ_2.

The linear mapping induced by the double coset operator $\Gamma_1\alpha\Gamma_2$ from $M_k(\Gamma_1)$ into $M_k(\Gamma_2)$ is called a **Hecke operator**.

2. Hecke Algebras

Note: This section is taken from [3].

Let Γ and Γ' be two congruence subgroups and let $\Delta \subset GL_2^+(\mathbb{Q})$ be a semigroup. We denote by $\mathfrak{R}(\Gamma,\Gamma';\Delta)$ the free $\mathbb{Z}-$ module generated by double cosets $\Gamma\alpha\Gamma'$ with $\alpha \in \Delta$ i.e.

$$\mathfrak{R}(\Gamma,\Gamma';\Delta) = \{\sum_{\alpha\in\Delta} a_\alpha \Gamma\alpha\Gamma' | a_\alpha \in \mathbb{Z} \text{ and } a_\alpha = 0 \text{ except for finitely many } \alpha\}.$$

One particular case of interest is when $\Gamma = \Gamma'$ and we write $\mathfrak{R}(\Gamma, \Delta) = \mathfrak{R}(\Gamma, \Gamma'; \Delta)$. In this section, we attempt to define multiplication of elements of $\mathfrak{R}(\Gamma, \Delta)$ so that $\mathfrak{R}(\Gamma, \Delta)$ becomes an algebra.

Let $\Gamma_1, \Gamma_2, \Gamma_3$ be congruence subgroups. For two elements $\Gamma_1\alpha\Gamma_2 = \sqcup_i \Gamma_1\alpha_i$ and $\Gamma_2\beta\Gamma_3 = \sqcup_j \Gamma_2\beta_j$, we define multiplication of $\Gamma_1\alpha\Gamma_2$ and $\Gamma_2\beta\Gamma_3$ by

$$\Gamma_1\alpha\Gamma_2 \cdot \Gamma_2\beta\Gamma_3 = \sum_\gamma c_\gamma \Gamma_1\gamma\Gamma_3, \tag{1}$$
$$c_\gamma = |\{(i,j)|\Gamma_1\alpha_i\beta_j = \Gamma_1\gamma\}|,$$

where the summation is taken over all double cosets $\Gamma_1\gamma\Gamma_3$ such that $\gamma \in \Delta$. The right hand side is a finite sum because there are only finitely many i's and j's. We can extend the multiplication linearly

Lemma 2.1. *The multiplication defined by Equation 1 is independent of the choice of the representatives α_i, β_j and γ.*

Proof. Let $\mathbb{Z}[\Gamma_1 \setminus \Delta]$ be the free $\mathbb{Z}-$module generated by left-cosets $\Gamma\alpha, (\alpha \in \Delta)$. The correspondence $\Gamma_1\alpha\Gamma_2 = \sqcup_i \Gamma_1\alpha_i$ to $\sum_i \Gamma_1\alpha_i$ induces an injective $\mathbb{Z}-$module homomorphism of $\mathfrak{R}(\Gamma_1, \Gamma_2; \Delta)$ into $\mathbb{Z}[\Gamma_1 \setminus \Delta]$. There is a right action of Γ_2 on $\mathbb{Z}[\Gamma_1 \setminus \Delta]$. Considering $\mathfrak{R}(\Gamma_1, \Gamma_2; \Delta)$ as a $\mathbb{Z}-$submodule of $\mathbb{Z}[\Gamma_1 \setminus \Delta]$, one then obtains that

$$\mathfrak{R}(\Gamma_1, \Gamma_2; \Delta) = \mathbb{Z}[\Gamma_1 \setminus \Delta]^{\Gamma_2}.$$

This is because if $\gamma_2 \in \Gamma_2$, then $\Gamma_1\alpha\Gamma_2 \cdot \gamma_2 = \Gamma_1\alpha\Gamma_2$ and hence $\mathfrak{R}(\Gamma_1, \Gamma_2; \Delta) \subset \mathbb{Z}[\Gamma_1 \setminus \Delta]^{\Gamma_2}$. Also if $\zeta = \sum_\alpha a_\alpha \Gamma_1\alpha \in \mathbb{Z}[\Gamma_1 \setminus \Delta]^{\Gamma_2}$, we want to prove that $\zeta \in \mathfrak{R}(\Gamma_1, \Gamma_2; \Delta)$. It suffices to show that if $a_\alpha \neq 0$ and if we write $\Gamma_1\alpha\Gamma_2 = \sqcup_i \Gamma_1\alpha\sigma_i$, where $\sigma_i \in \Gamma_2$, then $a_\alpha = a_{\alpha\sigma_i}$ for all the coset representatives σ_i of $\Gamma_1 \setminus \Gamma_1\alpha\Gamma_2$. This follows automatically since $\sigma_i \in \Gamma_2$ and hence $\zeta \cdot \sigma_i = \zeta$. Also since $\mathbb{Z}[\Gamma_1 \setminus \Delta]$ is a free $\mathbb{Z}-$module, this would imply that $a_{\alpha\sigma_i} = a_\alpha$. This gives us the reverse inclusion.

Now we can define an action of the double coset $\Gamma_2\beta\Gamma_3$ on $\mathbb{Z}[\Gamma_1 \setminus \Delta]^{\Gamma_2}$, for every $\beta \in \Delta$. First we write $\Gamma_2\beta\Gamma_3 = \sqcup_j \Gamma_2\beta_j$. The action of $\Gamma_2\beta\Gamma_3$ on $\zeta = \sum_\alpha a_\alpha \Gamma_1\alpha \in \mathbb{Z}[\Gamma_1 \setminus \Delta]^{\Gamma_2}$ is given as follows

$$\sum_\alpha a_\alpha \Gamma_1\alpha \cdot (\Gamma_2\beta\Gamma_3) = \sum_{\alpha,j} a_\alpha \Gamma_1\alpha\beta_j. \tag{2}$$

One can see that the action does not depend on the choice of representatives β_j by an argument similar to the one given in Definition 1.1. Writing $\Gamma_1\alpha\Gamma_2 = \sqcup_i \Gamma_1\alpha_i$ as an element of $\mathbb{Z}[\Gamma_1 \setminus \Delta]^{\Gamma_2}$, we see that

$$(\sum_i \Gamma_1\alpha_i) \cdot (\Gamma_2\beta\Gamma_3) = \sum_{i,j} \Gamma_1\alpha_i\beta_j. \tag{3}$$

If $\gamma_3 \in \Gamma_3$, then we know that $\beta_i\gamma_3$'s also form coset representatives for $\Gamma_2\beta\Gamma_3$ and this shows that $(\sum_i \Gamma_1\alpha_i) \cdot (\Gamma_2\beta\Gamma_3) \in \mathbb{Z}[\Gamma_1 \setminus \Delta]^{\Gamma_3} = \mathfrak{R}(\Gamma_1, \Gamma_3; \Delta)$. So we can write

$$(\Gamma_1\alpha\Gamma_2) \cdot (\Gamma_2\beta\Gamma_3) = \sum_{\gamma} b_\gamma \Gamma_1\gamma\Gamma_3. \tag{4}$$

Equation 3 shows us that the action defined in Equation 4 coincides with the one defined in Equation 1. Hence the multiplication does not depend on the choice of γ as well.

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The $\mathbb{Z}-$algebra $\mathfrak{R}(\Gamma, \Delta)$ is called a **Hecke algebra**.(We omit the details required to prove that multiplication is associative. It is not hard to prove it and the proof is along the lines of the proof of Lemma 2.1) The unity in $\mathbb{Z}-$algebra $\mathfrak{R}(\Gamma, \Delta)$ is Γ.

3. Hecke operators according to [3]

There are various approaches to define Hecke operators. We focus our attention on the definition given by Miyake in [3].

We define the following semigroups $\Delta_0(N)$ and $\Delta_1(N)$ of $GL_2^+(\mathbb{Q})$ as follows.

$$\Delta_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{Z}) : c \equiv 0(\bmod N), (a, N) = 1, ad - bc > 0 \right\}$$

$$\Delta_1(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{Z}) : c \equiv 0(\bmod N), a \equiv 1(\bmod N), ad - bc > 0 \right\}$$

One primarily studies the Hecke algebra $\mathfrak{R}(\Gamma_0(N), \Delta_0(N))$ and $\mathfrak{R}(\Gamma_1(N), \Delta_1(N))$.

$\Gamma_1(N)$ is a normal subgroup of $\Gamma_0(N)$ and hence there is a natural action of $\Gamma_0(N)$ on $M_k(\Gamma_1(N))$. The action is defined as follows. If $f \in M_k(\Gamma_1(N))$ and $\alpha \in \Gamma_0(N)$, then $f \to f|_k[\alpha]$. Also $\frac{\Gamma_0(N)}{\Gamma_1(N)} \cong (\frac{\mathbb{Z}}{N\mathbb{Z}})^\times$. Since $M_k(\Gamma_1(N))$ forms a $\mathbb{C}-$vector space and has an action of a finite abelian group $\frac{\Gamma_0(N)}{\Gamma_1(N)}$, we have the following direct sum decomposition

$$M_k(\Gamma_1(N)) = \bigoplus_{\chi} M_k(\Gamma_0(N), \chi) \tag{5}$$

where the summation runs over all Dirichlet characters mod N. We state the following useful theorem from Miyake [3] without proof.

Theorem 3.1. $\mathfrak{R}(\Gamma_1(N), \Delta_1(N))$ *is isomorphic to* $\mathfrak{R}(\Gamma_0(N), \Delta_0(N))$ *through the correspondence*

$$\Gamma_1(N)\alpha\Gamma_1(N) \longrightarrow \Gamma_0(N)\alpha\Gamma_0(N) \qquad (\alpha \in \Delta_1(N)). \tag{6}$$

Also the correspondence defined in 6 is commutative with the natural embedding of $M_k(\Gamma_0(N), \chi)$ *into* $M_k(\Gamma_1(N))$ *i.e. the following diagram is commutative*

$$\begin{array}{ccc} M_k(\Gamma_0(N), \chi) & \xrightarrow{f} & M_k(\Gamma_0(N), \chi) \\ \downarrow & & \downarrow \\ M_k(\Gamma_1(N)) & \xrightarrow{g} & M_k(\Gamma_1(N)) \end{array} \tag{7}$$

where f *is the linear map induced by* $\Gamma_0(N)\alpha\Gamma_0(N)$*,* g *is the linear map induced by* $\Gamma_1(N)\alpha\Gamma_1(N)$ *and the vertical arrows are inclusions.*

5 and 3.1 suggest that perhaps it is enough to study the Hecke algebra $\mathfrak{R}(\Gamma_0(N), \Delta_0(N))$ acting on $M_k(\Gamma_0(N), \chi)$. For the sake of simplicity, we look at Hecke operators on $M_k(\Gamma_0(N))$, i.e. take χ to be the trivial character.

We define elements $T(n)$ and $T(l, m)$ of $\mathfrak{R}(\Gamma_0(N), \Delta_0(N))$ (taken from [3]) by

$$T(l, m) = \Gamma_0(N) \begin{bmatrix} l & 0 \\ 0 & m \end{bmatrix} \Gamma_0(N) \tag{8}$$

$$T(n) = \sum_{det(\alpha)=n} \Gamma_0(N)\alpha\Gamma_0(N), \tag{9}$$

where the summation is taken over all the double cosets $\Gamma_0(N)\alpha\Gamma_0(N)$ in $\mathfrak{R}(\Gamma_0(N), \Delta_0(N))$ with $det(\alpha) = n$. These are the Hecke operators that are mainly discussed in [3].

4. Modular forms of level N as functions on "Modular points"

In this section, we follow an alternative viewpoint of Modular forms of level N as given in Lang's book [2]. We identify modular forms of level N with respect to $\Gamma_1(N)$ as homogeneous functions on "modular points" with some properties.

We consider pairs (t, L) where L is a lattice and t is a point on the elliptic curve $\frac{\mathbb{C}}{L}$, of exact order N. The set of all such pairs is called the **modular set** for $\Gamma_1(N)$. A pair (t, L) in the modular set is called a **modular point**. Let k be an integer. We denote by $\mathfrak{F}_1(N, k)$ as the vector space of functions F on modular points (t, L), satisfying the conditions

$$\text{F is homogeneous of degree } -k \text{ i.e.} \tag{10}$$
$$F(\lambda t, \lambda L) = \lambda^{-k} F(t, L), \qquad \forall \lambda \in \mathbb{C}^\times$$

Now consider the set $S_1(N, k)$ of functions f on the upper half plane $\mathbb{H}$, such that

$$f(\gamma\tau) = (c\tau + d)^k f(\tau), \tag{11}$$
$$\text{where } \tau \in \mathbb{H} \text{ and } \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_1(N).$$

Consider the map

$$\begin{aligned}\phi_1 : \mathfrak{F}_1(N,k) \to S_1(N,k) \\ \phi_1(F) = f, \\ f(\tau) = F([\frac{1}{N}], \mathbb{Z}\cdot\tau \oplus \mathbb{Z}\cdot 1)\end{aligned} \tag{12}$$

ϕ_1 is well defined because if $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_1(N)$, then for all $\tau \in \mathbb{H}$,

$$\begin{aligned} f(\gamma\tau) = F\left([\frac{1}{N}], \mathbb{Z}\cdot\frac{a\tau+b}{c\tau+d}\tau \oplus \mathbb{Z}\cdot 1\right) \\ = (c\tau+d)^k F\left(\frac{c\tau+d}{N}, \mathbb{Z}\cdot(a\tau+b) \oplus \mathbb{Z}\cdot(c\tau+d)\right) \\ = (c\tau+d)^k f(\tau)\end{aligned}$$

since $c \equiv 0(\text{mod } N), d \equiv 1(\text{mod } N)$ and $\mathbb{Z}\cdot(a\tau+b) \oplus \mathbb{Z}\cdot(c\tau+d) = \mathbb{Z}\cdot\tau \oplus \mathbb{Z}\cdot 1$.
Similarly one can define

$$\begin{aligned}\phi_2 : S_1(N,k) \to \mathfrak{F}_1(N,k) \\ \phi_2(g) = G, \\ G(t, \mathbb{Z}\cdot w_1 \oplus \mathbb{Z}w_2) = w_2^{-k} g(\frac{w_1}{w_2}).\end{aligned} \tag{13}$$

where w_2 is chosen so that $[\frac{w_2}{N}] = t$. Note that such a choice of w_2 is possible using the fact that the natural map $SL_2(Z) \longrightarrow SL_2\left(\frac{\mathbb{Z}}{N\mathbb{Z}}\right)$ is surjective. Also if we have another lattice $\mathbb{Z}v_1 \oplus \mathbb{Z}v_2 = \mathbb{Z}w_1 \oplus \mathbb{Z}w_2$ such that $\frac{v_2 - w_2}{N} \in \mathbb{Z}w_1 \oplus \mathbb{Z}w_2$, then one can write $v_1 = a\cdot w_1 + b\cdot w_2$ and $v_2 = c\cdot w_1 + d\cdot w_2$, for some $a,b,c,d \in \mathbb{Z}$. Also one notices that the change of basis matrix $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_1(N)$. So $G(t, \mathbb{Z}\cdot v_1 \oplus \mathbb{Z}v_2) = v_2^{-k} g(\frac{v_1}{v_2}) = v_2^{-k} g(\gamma\frac{w_1}{w_2}) = (v_2)^{-k}(c(\frac{w_1}{w_2}) + d)^k g(\frac{w_1}{w_2}) = G(t, \mathbb{Z}\cdot w_1 \oplus \mathbb{Z}w_2)$.

Finally to prove that ϕ_2 is well defined, one can verify that $G \in \mathfrak{F}_1(N,k)$ because

$$G(\lambda t, \mathbb{Z}\cdot\lambda w_1 \oplus \mathbb{Z}\lambda w_2)) = (\lambda w_2)^{-k} g\left(\frac{\lambda w_1}{\lambda w_2}\right) = \lambda^{-k} G(\mathbb{Z}\cdot w_1 \oplus \mathbb{Z}w_2)).$$

One can also check that ϕ_1 and ϕ_2 are inverses to each other. Under this bijection between $\mathfrak{F}_1(N,k)$ and $S_1(N,k)$ given by ϕ_1 and ϕ_2 , we obtain the following bijection

$$M_k(\Gamma_1(N)) \longleftrightarrow \mathfrak{F}(N,k) = \{g \in \mathfrak{F}_1(N,k) : \tau \to g(\mathbb{Z}\cdot\tau \oplus \mathbb{Z}\cdot 1) \text{ is holomorphic on } \mathfrak{h} \cup \infty\}. \tag{14}$$

5. Hecke operators as defined by [2]

Let $L_1(N)$ be the free vector space over $\mathbb{Q}$ defined by the modular points (t, L). For each positive integer n, we define an endomorphism T_n as follows

$$T_n : L_1(N) \longrightarrow L_1(N) \tag{15}$$
$$T_n : (t, L) \longrightarrow \frac{1}{n} \sum_{\substack{[L^{'}:L]=n \\ (t,L^{'})=N}} (t, L^{'}).$$

The condition $(t, L^{'}) = N$ means that t has an exact order N with respect to the elliptic curve $\frac{\mathbb{C}}{L^{'}}$. We now define how T_n "acts" on $M_k(\Gamma_1(N))$ using the correspondence given in equation 14. If $F \in \mathfrak{F}(N, k)$

$$T_n(F)(L) = \frac{1}{n} \sum_{\substack{[L^{'}:L]=n \\ (t,L^{'})=N}} F(t, L^{'}) \tag{16}$$

When we talk about Hecke operators of level 1, the modular points are simply lattices and we can view Hecke operators as endomorphisms of the free $\mathbb{Q}$ vector space over lattices. Serre in his book [5] gives the following definition of T_n as an endomorphism of the free $\mathbb{Z}$ algebra over the set of lattices:

$$T_n(L) = \sum_{[L:L^{'}]=n} L^{'} \tag{17}$$

As an operator on the set of modular forms of level 1,

$$T_n(F)(L) = n^{k-1} \sum_{[L:L^{'}]=n} F(L^{'}) \tag{18}$$

In [4], Ribet and Stein show that

$$T_n(F)(L) = n^{k-1} \sum_{[L:L^{'}]=n} F(L^{'}) = \frac{1}{n} \sum_{[L^{''}:L=n]} F(L^{''})$$

which agrees with 16.

One could also the alter the definitions of the action of Hecke operators on a lattice L and the space of modular forms as given in 15 and 16 in terms of lattices contained in L instead of lattices containing L as follows:

$$T_n((t, L)) = \sum_{\substack{[L:L^{''}]=n \\ (nt,L^{'})=N}} (nt, L^{''}). \tag{19}$$

and if $F \in M_k(\Gamma_1(N))$,

$$T_n(F)((t,L)) = n^{k-1} \sum_{\substack{[L:L^{''}]=n \\ (nt,L^{''})=N}} F(nt, L^{''}). \tag{20}$$

Let us suppose that we are given lattices L and $L^{'}$ such that $[L^{'} : L] = n$. Let $L^{''} = nL^{'}$. Then we have

$$[L : L^{''}] = n.$$
$$(t, L^{'}) = N \iff (nt, L^{''}) = N.$$

Using the fact that F is homogeneous of degree $-k$ we obtain that

$$T_n(F)((t,L)) = n^{k-1} \sum_{\substack{[L:L^{''}]=n \\ (nt,L^{''})=N}} F(nt, L^{''}) = \frac{1}{n} \sum_{\substack{[L^{'}:L]=n \\ (t,L^{'})=N}} F(t, L^{'})$$

This shows us that the definitions given in 19 and 20 agree well with the definitions given in 15 and 16 since the two different actions of T_n on $M_k(\Gamma_1(N))$ agree.

6. Equivalence between the two definitions

If $\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{Z})$ and if $f \in M_k(\Gamma_1(N))$, then

$$(f|_k[\alpha])(z) = det(\alpha)^{\frac{k}{2}}(cz+d)^{-k} f(\frac{az+b}{cz+d}). \tag{21}$$

Let us say that f corresponds to F which is a function on modular points, under the bijection given in 14. Let $L = \mathbb{Z} \cdot w_1 \oplus \mathbb{Z} \cdot w_2$ and $t = [\frac{w_2}{N}] \in \frac{\mathbb{C}}{L}$.

$$F(\alpha \cdot t, \alpha \cdot L) = F\left(\left[\frac{cw_1 + dw_2}{N}\right], \mathbb{Z} \cdot (aw_1 + bw_2) \oplus \mathbb{Z} \cdot (cw_1 + dw_2)\right) \tag{22}$$

$$= f\left(\frac{aw_1 + bw_2}{cw_1 + dw_2}\right)(cw_1 + dw_2)^{-k}. \tag{23}$$

If $\tau \in \mathbb{H}$, then let $i(\tau) = (\left[\frac{1}{N}\right], \mathbb{Z} \cdot \tau \oplus \mathbb{Z} \cdot 1)$. We identify the upper half plane with modular points through the map i. For the sake of simplicity, henceforth a modular point belongs to $i(\mathbb{H})$. If $F \in \mathfrak{F}(N)$ (where $\mathfrak{F}(N)$ denotes the space of continuous complex valued functions on modular points and by the assumption that a modular point belongs to $i(\mathbb{H})$, $\mathfrak{F}(N)$ can be identified with H^*), we can define an action of $M_2(\mathbb{Z})$ if we view $M_2(\mathbb{Z})$ acting on functions of modular points as follows:

$$F|_k[\alpha](t, L) = det(\alpha)^{\frac{k}{2}} F(t \cdot \alpha, L \cdot \alpha). \tag{24}$$

We have the following commutative diagram

$$\begin{tikzcd} S_1(N,k) \arrow[r, "\alpha"] \arrow[d] & \mathbb{H}^* \arrow[d] \\ \mathfrak{F}_1(N,k) \arrow[r, "\alpha"] & \mathfrak{F}(N) \end{tikzcd} \tag{25}$$

Here the vertical arrows are bijections and the horizontal arrows are defined by 21 and 24. Let $L = \{(u,v) : u, v \in \mathbb{Z}\}$ and $L_0 = \left\{(u, \frac{v}{N}) : u, v \in \mathbb{Z}\right\}$

To prove the equivalence of the two definitions given in 8 and 20, we need to prove the following (we fix level N and a natural number n)

- If $\alpha \in \Delta_1(N)$ and $det(\alpha) = n$, then $[L : L \cdot \alpha] = n$, and $(0, \frac{1}{N}) \cdot \alpha - (0, \frac{n}{N}) \in L \cdot \alpha$. And $(0, \frac{n}{N})$ is a point on $\frac{\mathbb{C}}{L'}$ of exact order N.
- If $L^{'}$ is a lattice such that $[L : L^{'}] = n$ and $(0, \frac{n}{N})$ is a point on $\frac{\mathbb{C}}{L'}$ of exact order N, then there exists $\alpha \in \Delta_1(N)$ such that $L \cdot \alpha = L^{'}$ and $(0, \frac{n}{N}) - (0, \frac{1}{N}) \cdot \alpha \in L^{'}$.
- If there exists another $\beta \in M_2(\mathbb{Z})$ such that $L \cdot \beta = L^{'}$ and $(0, \frac{n}{N}) - (0, \frac{1}{N}) \cdot \beta \in L^{'}$, then there exists γ_1 and $\gamma_2 \in \Gamma_1(N)$ such that $\beta = \gamma_1 \alpha \gamma_2$.
- Conversely, if α, β belong to the same double coset in $\Gamma_1(N) \setminus \Delta_1(N)/\Gamma_1(N)$, then $L \cdot \alpha = L \cdot \beta$ and $(0, \frac{1}{N})(\alpha - \beta) \in L \cdot \alpha$.

Lemma 6.1. *If $\alpha \in \Delta_1(N)$ and $det(\alpha) = n$, then $[L : L \cdot \alpha] = n$, and $(0, \frac{1}{N}) \cdot \alpha - (0, \frac{n}{N}) \in L \cdot \alpha$.*

Proof. Let $\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Delta_1(N)$. Clearly $[L : L \cdot \alpha] = n$. Also $(0, \frac{1}{N}) \cdot \alpha = (\frac{c}{N}, \frac{d}{N}))$. Then $(0, \frac{1}{N}) \cdot \alpha - (0, \frac{n}{N}) = (\frac{c}{N}, \frac{d-n}{N})$. We want to show that $(\frac{c}{N}, \frac{d-n}{N}) \in L \cdot \alpha$ or equivalently, there exists integers x, y such that

$$\left(\frac{c}{N}, \frac{d-n}{N}\right) = (x, y) \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{26}$$

Multiplying both sides by the inverse of α(on the right), we obtain the equivalent condition that there exists integers x, y such that

$$\frac{1}{n \cdot N}(nc, n \cdot (1-a)) = (x, y). \tag{27}$$

Since $c \equiv 0 (\text{mod } N)$ and $a \equiv 1 (\text{mod } N)$, this shows that we can find unique integers x, y satisfying equation 26 and this proves the lemma. Also noting that 27 and 26 are equivalent, we can conclude the next lemma. ■

Lemma 6.2. *Suppose $\alpha \in M_2(\mathbb{Z})$ and $det(\alpha) = n$. If $(0, \frac{1}{N}) \cdot \alpha - (0, \frac{n}{N}) \in L \cdot \alpha$, then $\alpha \in \Delta_1(N)$.*

Lemma 6.3. *Suppose $\alpha, \beta \in \Delta_1(N)$ such that $det(\alpha) = det(\beta) = n$ and $\gamma_1 \beta \gamma_2 = \alpha$ for some $\gamma_1, \gamma_2 \in \Gamma_1(N)$. Then $L \cdot \alpha = L \cdot \beta$ and $(0, \frac{1}{N})(\alpha - \beta) \in L \cdot \alpha$.*

Proof. Since $\gamma_1, \gamma_2 \in SL_2(\mathbb{Z})$, then $L \cdot \alpha = L \cdot (\gamma_1 \cdot \beta \gamma_2) = (L \cdot \gamma_1) \cdot (\beta \cdot \gamma_2) = (L \cdot \beta) \cdot \gamma_2 = L \cdot \beta$.

Also from lemma 6.1, $(0, \frac{1}{N}) - (0, \frac{n}{N}) \in L \cdot \alpha = L \cdot \beta \implies (0, \frac{1}{N})(\alpha - \beta) \in L \cdot \alpha$. ■

Lemma 6.4. *Suppose $\alpha, \beta \in \Delta_1(N)$ such that $det(\alpha) = det(\beta) = n$. Also suppose that $L \cdot \alpha = L \cdot \beta$ and $(0, \frac{1}{N})(\alpha - \beta) \in L \cdot \alpha$. Then there exists $\gamma_1, \gamma_2 \in \Gamma_1(N)$ such that $\gamma_1 \beta \gamma_2 = \alpha$.*

Proof. One first uses the following lemma (Lemma 4.5.2 from [3]):

Lemma 6.5. *If $\delta \in \Delta_0(N)$, then there exists positive integers l and m such that $(l, N) = 1$,$l|m$ and*

$$\Gamma_0(N)\alpha\Gamma_0(N) = \Gamma_0(N) \begin{bmatrix} l & 0 \\ 0 & m \end{bmatrix} \Gamma_0(N) \tag{28}$$

The pair (l, m) is uniquely determined by the lattice $L \cdot \alpha$.

Since we have that $L \cdot \alpha = L \cdot \beta$, using 6.5 we obtain that $\alpha \sim \beta$ in $\Gamma_0(N) \setminus \Delta_1(N)/\Gamma_0(N)$. Also from the proof of theorem 4.5.18 in [3], one obtains that since $\alpha \in \Delta_1(N)$, $\Gamma_0(N)\alpha\Gamma_0(N) = \Gamma_0(N)\alpha\Gamma_1(N)$. Thus there exists $\gamma_3 \in \Gamma_0(N)$ and $\gamma_4 \in \Gamma_1(N)$ such that $\beta = \gamma_3\alpha\gamma_4$. Since $\alpha\gamma_4 \in \Delta_1(N)$ and $\gamma_3 \in \Gamma_0(N)$, one finds that $\gamma_3 \in \Gamma_1(N)$ and hence $\alpha \sim \beta$ in $\Gamma_1(N) \setminus \Delta_1(N)/\Gamma_1(N)$.

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Remark 1 : Lemma 4.5.2 and Theorem 4.5.18 are taken from the book by Miyake [3]. It is not tough to follow the proof from [3] but I found it tedious to reproduce the proofs here.

Remark 2: Lemmas 6.1,6.2,6.3 and 6.4 prove the equivalence of the two definitions given in [2] and [3].

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